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A
TREATISE
OF
ALGEBRA,

In TWO BOOKS.

BOOK I.

CONTAINING,
The Fundamental Principles of this ART.
Together with
All the Practical Rules of OPERATION.

BOOK II.

CONTAINING,
A great VARIETY of PROBLEMS,
In the moſt important
BRANCHES of the MATHEMATICS.

Vix quicquam in univerſa Matheſi eſt ita difficile aut arduum occurrere poſſe, quò non inoffenſo pede per hanc methodum penetrare liceat.

SCHOOT. Pref. to DES CARTES.

THE SECOND EDITION.

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From Chas. F. Adams,
July 2, 1891.

THE

PREFACE.

THE *ſubject of the following book is* Algebra, *a ſcience of univerſal uſe in the Mathematics. Its buſineſs and uſe is to ſolve difficult problems, to find out rules and theorems in any particular branch of ſcience; to diſcover the properties of ſuch quantities as are concerned in any ſubject we have a mind to conſider. It properly follows theſe two fundamental branches,* Arithmetic *and* Geometry, *but is vaſtly ſuperior in nature to both, as it can ſolve queſtions quite beyond the reach of either of them.*

THIS *is an art truly ſublime, and of an unlimited extent; for if the conditions of a problem be never ſo complex, and though the quantities concerned are never ſo much entangled with one another, yet the Algebraiſt can find means to diſſolve and ſeparate them; or if they be ever ſo remote, his art can furniſh him with methods to bring them together and compare them. It is true he is often obliged to traverſe by many round-about ways, to get the relation of the quantities concerned; yet by certain rules he can purſue the computation of his problem through all theſe intricate turnings and windings; and by his ſkill and* ſagacity *can* hunt *it through all theſe labyrinths, till he arrives ſafely*

at

at the end of the chace, viz. *the ſolution of the problem.*

The *extent of this curious art is ſo great that it has gained the title of* Univerſal Mathematics; *and is called by way of eminence,* The Great Art; *and has been eſteemed the very apex of human reaſon. It is alſo called* Specious Arithmetic, Univerſal Arithmetic, The Analytic Art, The Art of Reſolution and Equation; *with a view to ſome or other of its properties or operations.*

The *nature of this excellent art is ſuch, that it may be applied to any ſubject, provided the principles of that ſubject, it is applied to, be underſtood. Its great beauty is, that it deals in* generals. *For whilſt other branches go no farther than their own particular ſubject, and can only find ſolutions in particular caſes; this art finds out general ſolutions, general rules, general theorems, and general methods.*

This *noble ſcience has alſo this peculiar property, that it not only inveſtigates rules in all the other parts of the Mathematics; but by the moſt ſubtle art and invention, it finds out its own rules, models them according to any form, and varies them at pleaſure, ſo as to anſwer any end propoſed. It would be in vain to attempt to enumerate all the uſes of this admirable art.*

By *making uſe of letters inſtead of numbers, it has one great advantage above arithmetic,* viz. *that in the ſeveral operations of arithmetic, the numbers are loſt or ſwallowed up, and changed into others: but here they are preſerved diſtinct, viſible, and unchanged. By which means general rules are drawn from particular ſolutions, to anſwer all caſes of like nature.*

By *help of algebraic characters, geometrical demonſtrations are often rendered more ſhort, compendious and clear. So that by this means we avoid the tediousneſs of a long verbal proceſs, which otherwiſe we ſhould neceſſarily be involved in; and which never fails to darken and obſcure the ſubject.*

It *is highly probable the ancients made uſe of ſome ſort of analyſis, whereby they found out their noble theories. For it is hardly poſſible ſo many fine theorems in Geometry, ſhould be groped out or ſtumbled on, without ſome ſuch method. But as it was then only in its infancy, it muſt have been far ſhort of the perfection we have it in at preſent.*

As *to the Reader's qualifications, it is abſolutely neceſſary that he underſtand Arithmetic and Geometry, as the keys to all the reſt. And it is alſo neceſſary that he underſtand the principles of every branch of ſcience, to which he would apply algebraic calculations; otherwiſe it would be in vain to attempt the ſolution of any problems therein, by the help of Algebra.*

Then *as to the method I have followed, it is this. I have gathered together the moſt valuable rules and precepts, which lie ſcattered up and down in all the beſt books of Algebra; and what was deficient, I have ſupplied as well as I could. Then I have thrown all theſe precepts and rules of working, into ſo many problems; which I have reduced into as ſhort a compaſs, and expreſſed in as plain terms as poſſible, ſo as they may be clear and intelligible. And the method I have taken I ſuppoſe will appear to be very ſimple and eaſy, and will readily be apprehended by ſuch people as have found confuſion and difficulty in other methods. I believe I have omitted nothing that is fundamental; and* *if any thing of leſs moment is paſſed by, it is either becauſe it is of little uſe, or is ſupplied by ſome other method or rule. And all the rules and problems are in ſuch order, that the eaſieſt appear firſt, and lead on to the harder, which follow in due courſe afterwards: theſe make up the firſt book. And the ſecond book contains the application of Algebra to all ſorts of problems, of which there is great variety, and many of them perfectly new; others that are not ſo, have generally new ſolutions to them. So I hope I have delivered both the principles and the practice at large, and yet have not clogged the Reader with any ſuperfluity.*

W. Emerſon.

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ALGEBRA.

DEFINITIONS.

1. *ALGEBRA* is a general method of computing Problems, by help of the letters of the alphabet, and other characters. It is of the ſame nature as Arithmetic, but more general, and therefore it is called *Univerſal Arithmetic*, as likewiſe the *Analytic Art.* The peculiar practice of this method is, to aſſume the quantity ſought as if it was known, and proceeding to work by the rules of this art, till at laſt the quantity ſought, or ſome powers thereof, be found equal to ſome given quantity, and conſequently itſelf becomes known.

2. *Like quantities*, are thoſe that conſiſt of the ſame letters; as a, $4a$, $-3a$. Alſo bb, $3bb$, $-11bb$; alſo $2abc$, $15abc$, $-abc$; &c.

3. *Unlike quantities*, are thoſe conſiſting of different letters, or of the ſame letters, differently repeated. As a, b, $2c$, $-3d$. Alſo a, $2aa$, $-5aaa$.

4. *Given quantities*, are thoſe whoſe values are known.

5. *Unknown quantities*, are thoſe whoſe values are not known.

6. *Simple quantities*, are thoſe conſiſting of one term only; as $5b$, $3a^2c$, $13dcc$, &c.

7. *Compound quantities*, are thoſe conſiſting of ſeveral terms, as $a+b$, $2a-3c$, $a+2b-3d$, &c.

8. *Poſitive quantities*, are thoſe to be added.

9. *Negative quantities*, are thoſe to be ſubtracted.

10. *Like ſigns*, are either all +, or all −, (See the Characters.)

11. *Unlike ſigns* are + and −.

12. *The Coefficient*, is the number prefixed to any letter or letters in any term. As 3 is the coefficient of $3aa$. If no number be prefixed, then 1 muſt be underſtood, as aa ſignifies $1aa$.

13. *A Binomial quantity*, is one conſiſting *of two terms*, as $2a + 3b$. A *Trinomial* of 3 terms, as $a + b - c$. A *Quadrinomial* of four, *&c.* A *Reſidual* is a binomial, where one of the quantities is negative.

14. *Power* of a quantity, is its ſquare, cube, biquadrate, *&c.*

15. *An Equation*, is the mutual comparing of one thing with another, by the ſign of equality put between them.

16. *A dependent Equation*, is an equation which may be deduced from ſome others.

17. *An independent Equation*, is one that cannot, by any means, be produced from the others.

18. *Pure Equation*, is an equation containing but one power of the unknown quantity; as *a ſimple Equation*, a *pure Quadratic*, a *pure Cubic*, &c.

19. *An affected Equation*, is that which contains ſeveral powers of the unknown quantity; and is denominated according to the higheſt power in it; as an *affected Quadratic*; an *affected Cubic*; an *affected fourth Power*; &c. Thus a ſimple equation contains only the ſimple quantity itſelf. A quadratic, a quantity of 2 dimenſions; a cubic, a quantity of 3 dimenſions; a biquadratic, of 4 dimenſions; *&c.*

20. *Index* or *Exponent*, is the number ſet over a letter, ſhewing what power it is: as a^3; here 3

ſhews it is the third power; or that a^3 is equivalent to aaa. And thus a^4 is the ſame as $aaaa$; a^5 the ſame as $aaaaa$; &c. the index always ſhewing how oft the letter is repeated.

21. *A Fraction*, conſiſts of two quantities placed one above another, with a line between them, as $\frac{a}{b}$; the upper (a) is called the *numerator*, the lower (b) the *denominator*.

22. *A Surd*, is a quantity that has not a proper root, as ſquare root of a ($\sqrt{a}$), cube root of bb ($\sqrt[3]{bb}$), &c. Roots of compound quantities that contain other ſurds, are called *Univerſal Surds*.

23. *A rational quantity*, is a quantity that has no radical ſign.

Characters uſed in Algebra.

$+$ *more*, *to be added*, being the ſign of addition. This is called an affirmative ſign. Thus $a + b$ ſignifies b added to a.

$-$ *leſs*, *abating*, the ſign of ſubtraction. This is alſo called a negative ſign. Thus $a - b$ ſignifies b ſubtracted from a.

Theſe ſigns always affect the quantity following; and are always to be interpreted in a contrary ſignification. If $+$ ſignifies *upward*, *forward*, *gain*, *increaſe*, *above*, *before*, *addition*, &c. then $-$ is to be interpreted *downward*, *backward*, *loſs*, *decreaſe*, *below*, *behind*, *ſubtraction*, &c. And if $+$ be underſtood of theſe, then $-$ is to be interpreted of the contrary.

$\backsim$	*difference*; as $a \backsim b$, ſignifies the difference between a and b.
$\times$	*multiplied by*; as $a \times b$, ſignifies a multiplied by b. Likewiſe ab, ſignifies a multiplied by b. All letters joined together ſignifies a multiplication. For brevity's ſake points are often uſed inſtead of $\times$, as $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$, ſignifies $n \times \frac{n-1}{2} \times \frac{n-2}{3}$.
$\div$	*divided by*, as $a \div b$, ſignifies a divided by b. And $\frac{a}{b}$ ſignifies the ſame.
$=$	*equal to*, as $a + b = 2d$, ſignifies a and b equal to $2d$.
$\sqsubset$	*greater than*, as $a \sqsubset b$, is a greater than b.
$\sqsubset$	*leſſer than*, as $a \sqsubset b$, is a leſs than b.
$\surd$	*a root*, as $\sqrt{a}$, is ſquare root of a; $\sqrt[3]{a}$, cube root of a; $\sqrt[4]{a}$, fourth root of a; &c. It is called a *Radical Sign.*
⊙	*involved to*: as ⊙ 2, involved to the ſquare; ⊙ 3 involved to the cube; *&c.*
lω	*extracted*: lω 2, ſquare root; lω 3, cube root; *&c.*
$\overline{a+b+c}$,	*a line*, or *vinculum*, drawn over ſeveral quantities a, b, c, denotes that compound quantity to be conſidered jointly as a ſimple quantity.

EXPLANATION.

$aa - bb + 3cd$, ſignifies bb ſubtracted from aa, and $3cd$ added.

$\overline{aa - bb} - \overline{cd - dd}$, ſignifies, that $cc - dd$ is ſubtracted from $aa - bb$.

$\overline{aa + 2ab}$

$\overline{aa+2ab} \backsim \overline{rr-ss}$, ſignifies the difference between $aa+2ab$ and $rr-ss$.

$abcc$ ſignifies the product of a and b and cc.

$\overline{a+b}\times aa$, ſignifies the ſum of $a+b$ multiplied by aa.

$a+b\times aa$, ſignifies the product of b into aa is to be added to a.

$\overline{aa-2ab}^{2}$, ſignifies the ſquare of the compound quantity $aa-2ab$.

$\sqrt{bb+cc}$ ſignifies the ſquare root of $bb+cc$.

$\sqrt[3]{2ab-cc}$, ſignifies the cube root of $2ab-cc$.

$\frac{aa}{a-b}$, ſignifies aa divided by $a-b$.

$\sqrt{\frac{a^3}{xx-aa}}$ ſignifies the ſquare root of a^3 divided by $xx-aa$.

a^3b^2 ſignifies $aaa\times bb$, or the cube of a multiplied by the ſquare of b.

$\overline{3ax-xx}\sqrt{5ax}$, ſignifies the ſquare root of $5ax$ multiplied by $3ax-xx$; and ſo of others.

Quantities that have no ſign prefixed, muſt be underſtood to have the ſign $+$; leading quantities ſeldom have the ſigns put down, when they are affirmative.

If A B and C D be two lines; then A B × C D, in a geometrical ſenſe, ſignifies the rectangle made by the lines A B and C D.

Alſo $\frac{AB}{CD}$, ſignifies the ratio that A B has to C D.

NOTATION.

1. In the computation of problems, put the firſt letters of the alphabet, *b*, *c*, *d*, *f*, *g*, *h*, &c. for known quantities, and the laſt letters of the alphabet for unknown ones. Yet ſome put vowels for

unknown quantities, and the reſt of the alphabet for known ones.

2. For general forms, put the capitals A, B, C, D, *&c.* for the general quantities.

3. Or in univerſal forms, let the quantities be denoted by the Greek capitals, Γ, Δ, Ζ, Θ, Λ, Π, Σ, Υ, Φ, Ψ, Ω, and indices, coefficients, *&c.* by the ſmall letters, δ, ε, η, θ, λ, μ, ν, π, τ, φ.

4. In caſe of neceſſity, make uſe of any other ſort of letters, or of any characters, that have names, as ♄, ♃, ♂, ☉, ♀, ☿, ☽, ♉, ♌, ♐, ♓, &c.

AXIOMS.

1. If equal quantities be added to equal quantities, the ſums will be equal.

2. If equal quantities be taken from equal quantities, the remainders will be equal.

3. If equal quantities be multiplied by equal quantities, the products will be equal.

4. If equal quantities be divided by equal quanties, the quotients will be equal.

5. The equal powers or roots of equal quantities, are equal.

6. If to or from equal quantities, unequal ones be added or ſubtracted; the ſums or remainders will be unequal.

7. If equal quantities be multiplied or divided by unequal quantities; the products or quotients will be unequal.

8. Quantities ſeverally equal to a third, are equal to one another.

9. The whole is equal to all the parts taken together.

10. If a quantity be added, and the ſame quantity ſubtracted, they deſtroy one another, and are both reduced to nothing.

BOOK I.

The fundamental Principles of Algebra.

SECT. I.

The primary Operations of Algebra in Integers.

PROBLEM I.

To add ſeveral Quantities together.

1 RULE.

IF the quantities are like and have like ſigns; add all the coefficients together, for the coefficient to that quantity, and prefix the ſame ſign.

Ex. 1.

$$\begin{array}{rr} \text{to} & +5a \\ \text{add} & +7a \\ \hline \text{Sum} & +12a \end{array} \qquad \begin{array}{rr} \text{to} & -16ab \\ \text{add} & -5ab \\ \text{add} & -2ab \\ \hline & -23ab \end{array} \qquad \begin{array}{rrr} \text{to} & +4a & -3x \\ \text{add} & +5a & -x \\ \text{add} & +a & -5x \\ \hline \text{Sum} & +10a & -9x \end{array}$$

Ex. 2.

$$\begin{array}{rr} \text{to} & +135ab^2 \\ \text{add} & +17abb \\ & +3abb \\ & +abb \\ \hline \text{Sum} & +156abb \end{array} \qquad \begin{array}{r} -202xxy^3 \\ -105xxy^3 \\ -17xxy^3 \\ \hline -324xxy^3 \end{array}$$

2 RULE.

In like quantities with unlike ſigns; add all the affirmative coefficients, into one ſum; and all the negative ones into another; ſubtract the leſſer ſum from the greater, and to the difference prefix the ſign of the greater, with the proper quantity.

Ex. 3.

$$\begin{array}{rr} \text{to} & +6a \\ \text{add} & -3a \\ \hline \text{Sum} & +3a \end{array} \qquad \begin{array}{r} -16d \\ +3d \\ \hline -13d \end{array} \qquad \begin{array}{rr} 3a & -7b \\ -3a & +8b \\ \hline 0 & +b \end{array}$$

Ex. 4.

$$\begin{array}{rr} & -125ab \\ & -37ab \\ & +ab \\ & +99ab \\ \hline & -162ab \\ & +100ab \\ \hline \text{Sum} & -62ab \end{array} \qquad \begin{array}{r} +34x^2y \\ -8x^2y \\ -x^2y \\ +92x^2y \\ -67x^2y \\ \hline +126x^2y \\ -76x^2y \\ \hline +50x^2y \end{array}$$

Ex. 5.

$$\begin{array}{lrrrr} & -2aa & -9bcd & +dd & +2e \\ & +7aa & -20bcd & -dd & +5e \\ & +3aa & +4bcd & & \\ \hline \text{Sum} & +8aa & -25bcd & & +7e \end{array}$$

3 RULE.

Set down all the unlike quantities with their proper ſigns.

Ex. 6.

$$\begin{array}{r} +2a \\ +3b \\ -c \\ +d \\ \hline \end{array}$$

$$\text{Sum } 2a + 3b - c + d$$

Ex. 7.

$$\begin{array}{rrl}
 & +13aa & -2a \\
 & -4ab & -3a \\
 & +bc & +5a \\
 & -2dd & +6d \\
\hline
\text{Sum} & \multicolumn{2}{l}{+13aa-4ab+bc-2dd+6d.} \\
\hline
\end{array}$$

Ex. 8.

$$\begin{array}{rrrrr}
 & 2ee & +3ef & -ff & +17 \\
 & -3ee & +5ef & +2ff & -11 \\
 & +6ee & -ef & +ff & -3 \\
\hline
\text{Sum} & +5ee & +7ef & +2ff & +3 \\
\hline
\end{array}$$

The reaſon of this rule is evident for like ſigns; and in unlike ſigns, it follows from the nature of affirmative and negative quantities, that the difference ought to be taken, to make up the total. As if a man owes 10 *l.* then 10 *l.* ought to be deducted from his ſtock to find his real worth.

Cor. 1. *When ſeveral quantities are to be added together, it is the ſame thing, in whatever order they are placed.*

Thus $a+b-c=a-c+b=-c+a+b$ $=b+a-c$, &c. for all theſe are the ſame.

Cor. 2. *Hence the ſum of any number of affirmative quantities, is affirmative; and the ſum of any number of negative quantities, is negative.*

PROBLEM II.

To ſubtract quantities from one another.

RULE

Change the ſigns of all the quantities to be ſubtracted; and then add them all together by Prob. I. and the ſum will be the remainder ſought.

Ex. 1.

$$\begin{array}{rcr} +8a & 16b & -11c \\ +3a & -5b & +3c \\ \hline \text{Rem. } 8a-3a & 16b+5b & -11c-3c \\ \text{or } \quad 5a & \text{or } 21b & \text{or } -14c \\ \hline \end{array}$$

$$\begin{array}{r} -20 \\ -6 \\ \hline -20+6 \\ \text{or } -14 \\ \hline \end{array}$$

Ex. 2.

$$\begin{array}{lr} \text{from} & 6a-3x+6y-7 \\ \text{take} & +8a+4x+6y+5 \\ \hline \text{Rem.} & -2a-7x+0-12 \\ \hline \end{array}$$

Ex. 3.

$$\begin{array}{lrr} \text{from} & a+b & a+b \\ \text{take} & a-b & -a+b \\ \hline \text{Rem.} & +2b & 2a \end{array}$$

Ex. 4.

$$\begin{array}{lr} \text{from} & aa+2ab+bb \\ \text{take} & +4ab \\ \hline \text{Rem} & aa-2ab+bb \\ \hline \end{array}$$

Ex. 5.

$$\begin{array}{lr} \text{from} & aa-bb \\ \text{take} & cc-dd \\ \hline \text{Rem.} & aa-bb-cc+dd \\ \hline \end{array}$$

Ex. 6.

$$\begin{array}{lr} \text{from} & 3aa-2a+cd-dd-ff \\ \text{take} & -2aa-5a-ab-2dd \\ \hline \text{Rem.} & 5aa+3a+cd+ab+dd-ff \end{array}$$

Cor. *Hence, To ſubtract one quantity from another, is the ſame thing as to add them together, when all the ſigns of the ſubtrahend are changed.*

$a - - b = a + b.$

For it is the ſame thing to ſubtract —, as to add + ; and to add —, as to ſubtract +. For ſuppoſe a man to owe 10*l*; becauſe it is a debt it muſt be writ — 10*l.* therefore if any body would take away this — 10, it is the ſame thing as if he added + 10 to his ſtock: but before it is diſcharged, this — 10 is the ſame, as + 10 deducted out of his ſtock.

PROBLEM III.

To multiply one quantity by another.

RULE.

Multiply every particular term (or ſimple quantity) of the multiplier, into every term of the multiplicand, one after another; ſo that the coefficients be multiplied into the coefficients; and the letters into the letters, by placing them all together, like letters in a word. And prefix + to products of like ſigns, and — to unlike ones. The ſum of all is the product ſought.

Ex. 1.

$$\begin{array}{r} +a \\ +b \\ \hline +ab \end{array} \qquad \begin{array}{r} -a \\ -b \\ \hline +ab \end{array} \qquad \begin{array}{r} +3a \\ -2b \\ \hline -6ab \end{array} \qquad \begin{array}{r} -4c \\ +5d \\ \hline -20cd \end{array}$$

Ex. 2.

$$\begin{array}{l} a+b \\ a+b \\ \hline aa+ab \\ \qquad +ab+bb \\ \hline aa+2ab+bb \end{array} \qquad\qquad \begin{array}{l} a+b \\ a-b \\ \hline aa+ab \\ \qquad -ab-bb \\ \hline aa-bb \end{array}$$

Ex. 3.

$$\begin{array}{rrr}
3a & -2b & \\
5a & +4b & \\
\hline
15aa & -10ab & \\
 & +12ab & -8bb \\
\hline
15aa & +2ab & -8bb \\
\hline
\end{array}$$

Ex. 4.

$$\begin{array}{rrrr}
aa & +ab & -bb & \\
a & -b & & \\
\hline
a^3 & +aab & -abb & \\
 & -aab & -abb & +b^3 \\
\hline
a^3 & & -2abb & +b^3 \\
\hline
\end{array}$$

Ex. 5.

$$\begin{array}{l}
ab - 3cd + rs \\
5r - 7d \\
\hline
5rab - 15rcd + 5rrs - 7abd + 21cdd - 7rsd \\
\hline
\end{array}$$

Ex. 6.

$$\begin{array}{rrrrrr}
3aa & -2ab & +5 & & & \\
aa & +2ab & -3 & & & \\
\hline
3a^4 & -2ba^3 & +5aa & & & \\
 & +6ba^3 & -4aabb & +10ab & & \\
 & & -9aa & +6ab & -15 & \\
\hline
3a^4 & +4ba^3 & -4bbaa & -4aa & +16ab & -15 \\
\hline
\end{array}$$

Ex. 7.

$$\begin{array}{rrrr}
aa & +bb & & \\
cc & -dd & & \\
\hline
ccaa & +ccbb & & \\
-ddaa & -ddbb & & \\
\hline
ccaa & -ddaa & +ccbb & -ddbb \\
\end{array}$$

Ex. 8.

$$\begin{array}{c} a^3 \\ a^2 \\ \hline a^5 \end{array} \qquad\qquad \begin{array}{r} 4b^3 \\ -3b^5 \\ \hline -12b^8 \end{array}$$

That every term in the multiplicand muſt be multiplied by every term in the multiplier, is thus made evident. Let $a+b$ be multiplied by $c+d$; it is plain, $a+b$ muſt be taken ſo often as there are ſuppoſed units in c and d, that is, as often as there are units in c, and alſo as oft as there are units in d. Therefore the product will be $\overline{a+b}\times c+\overline{a+b}\times d$. But for the ſame reaſon $\overline{a+b}\times c=ac+bc$, alſo $\overline{a+b}\times d=ad+bd$. Whence the product will be $ac+bc+ad+bd$; that is, the ſum of all the products of every term multiplied by every term.

That like ſigns give $+$, and unlike ſigns $-$, in the product, will appear thus.

Caſe 1. Let $+a$ be multiplied by $+b$. Then ſince this multiplication ſuppoſes, that $+a$ is to be ſo often added together as there are units in $+b$; and the ſum of any number of affirmatives is affirmative, therefore the whole ſum is affirmative, that is $+a\times+b=+ab$.

Caſe 2. Let $+a$ be multiplied by $-b$. Now ſince this implies that $+a$ is to be as often ſubtracted as there are units in b; and the ſum of any number of negatives, is negative, therefore that whole ſum, is negative, that is, $+a\times-b=-ab$.

Caſe 3. Let $-a$ be multiplied by $+b$. It is plain here, that $-a$ is to be ſo often taken as there are units in b; and the ſum of any number of negatives being negative, therefore the whole ſum is negative; that is, $-a\times+b=-ab$.

Otherwiſe, Let $d-a$ be multiplied by $+b$; then (*Caſe* 1.) the product will be bd together with $-a\times+b$: but bd is too big, as being the product of d by b, inſtead of $\overline{d-a}$ by b ($d-a$ being leſs than d); therefore bd, being too much, the product $-a\times+b$ muſt be ſubtracted; that is, the true product will be $db-ab$; and conſequently $-ab=-a\times+b$.

Caſe 4. Let $-a$ be multiplied by $-b$. Here $-a$ is to be ſubtracted as often as there are units in b: but ſubtracting negatives is the ſame as adding affirmatives (*Cor. Prob.* 2.); conſequently the product is $+ab$.

Or thus. Since $a-a=0$, therefore $\overline{a-a}\times-b=0$, becauſe 0 multiplied by any thing produces 0; therefore ſince $\overline{+a-a}\times-b=0$; and the firſt term of the product is $-ab$ (*Caſe* 2); therefore the laſt term of the product muſt be $+ab$, to make the ſum 0, or $-ab+ab=0$; that is, $-a\times-b=+ab$.

Otherwiſe. Let $d-a$ be multiplied by $-b$. Then (*Caſe* 2.) the product will be $-bd$ together with $-a\times-b$; but $-bd$ the quantity to be ſubtracted is too big, being the product of d by $-b$, inſtead of $\overline{d-a}$ by $-b$, ($d-a$ being leſs than a); therefore the quantity $-bd$ to be ſubtracted being too much, ſomething muſt be reſtored, that is $-a\times-b$ muſt be added; and the true product will be $-bd+ab$; and therefore $+ab=-a\times-b$.

Cor. 1. *If ſeveral quantities are to be multiplied together; it is the ſame thing in whatever order it be done. Thus* $abc=acb=cab=bca$, &c. *for all theſe are equal.*

Cor. 2. *The powers of the ſame quantity are multiplied together, by adding their indices. Thus* $a^2\times a^3=a^{2+3}=a^5$.

Cor. 3. *Any odd number of —, multiplied together produce —; and any even number of —, produce +.*

SCHOLIUM.

In the multiplication of compound quantities, it is the beſt way to ſet them down in order, according to the dimenſions of ſome of the quantities. And in multiplying them, begin at the left hand, and multiply from the left hand towards the right, the way we write, which is contrary to the way we multiply numbers. But this will be moſt expeditious, and the ſeveral products will by this means be ſo ranged under one another, that like quantities will fall in the ſame places, which is the eaſieſt way for adding them up together.

In many caſes, the multiplication of compound quantities is only to be performed by writing their ſums, each under a vinculum, and putting the ſign (×) of multiplication between. As if the ſquare of $aa - xx$ was to be multiplied by $ag - bb$, and that by $ac + bd$, it may be written thus, $\overline{ax - xx}^2 \times \overline{ag - bb} \times \overline{ac + bd}$.

PROBLEM IV.

To divide one quantity by another.

1 RULE.

In ſimple quantities, which will divide without a remainder; divide the number by the number, and put the anſwer in the quotient. Then throw out all the letters in the dividend which are found in the diviſor, and place the remaining letters in the quotient. And like ſigns produce +, and unlike ſigns —, in the quotient.

Ex. 1.

$$\begin{array}{c} aa)\,aab\,(b \\ aab \\ \hline 0 \end{array} \qquad \begin{array}{c} 3ab)\,15aabcd\,(5acd \\ 15aabcd \\ \hline 0 \end{array}$$

Ex. 2.

$$\begin{array}{c} -aa)\,aab\,(-b \\ aab \\ \hline 0 \end{array} \;\Bigg|\; \begin{array}{c} -3ab)-15aabcd\,(5acd \\ -15aabcd \\ \hline 0 \end{array}$$

Ex. 3.

$$\begin{array}{c} -3cdd)-6ccdd\,(+2c \\ -6ccdd \\ \hline 0 \end{array}$$

Ex. 4.

$$\begin{array}{c} +6a^2b^2)-18b^3a^3d^3c\,(-3abcd^3 \\ -18b^3a^3d^3c \\ \hline 0 \end{array}$$

Ex. 5.

$$\begin{array}{c} -5a^2b)\,10a^2bbd\,(-2bd \\ 10a^2bbd \\ \hline 0 \end{array}$$

Ex. 6.

$$\begin{array}{c} 9x^2y)-9x^2y^2b\,(-yb \\ -9x^2y^2b \\ \hline 0 \end{array}$$

Ex. 7.

$$-8xx)\ -16x^3\ (+2x$$
$$-16x^3$$
$$\overline{\qquad 0 \qquad}$$

2 RULE.

In compound quantities, range the terms of the diviſor and dividend, according to the dimenſions of ſome letter. Then, by Rule 1, divide the firſt term of the dividend by the firſt term of the diviſor, placing the reſult in the quotient. Multiply the whole diviſor by the quotient, and ſubtract it from the dividend, to which bring down the next term of the dividend, call this the *Dividual.*

Divide the firſt term of the dividual by the firſt term of the diviſor; then multiply and ſubtract as before, and repeat the ſame proceſs till all the quantities be brought down. This is in effect the very ſame rule as is uſed in arithmetic.

Ex. 8.

$$a)\ ab + ac - a\ (b + c - 1 \text{ the quotient}$$
$$\underline{ab}$$
$$+ac$$
$$\underline{+ac}$$
$$-a$$
$$\underline{-a}$$
$$0$$

Ex. 9.

$$2b - 3c)\ 2baa - 3caa\ (aa$$
$$\underline{2baa - 3caa}$$
$$0$$

Ex.

Ex. 10.

$$\begin{array}{l}
a+b)\;ac+bc+ad+bd\;(c+d\\
\qquad\;\; \underline{ac+bc}\\
\qquad\qquad\qquad\;\; +ad+bd\\
\qquad\qquad\qquad\;\; \underline{+ad+bd}
\end{array}$$

Ex. 11.

$$\begin{array}{l}
yy-4)\;y^3-3yy-4y+12\;(y-3\\
\qquad\quad \underline{y^3\qquad\quad -4y\qquad\quad}\\
\qquad\quad -3yy\qquad\quad +12\\
\qquad\quad \underline{-3yy\qquad\quad +12}\\
\qquad\qquad\qquad 0
\end{array}$$

Ex. 12.

$$\begin{array}{l}
a+b)\;aa-bb\;(a-b\\
\qquad\quad \underline{aa+ab}\\
\qquad\qquad -ab-bb\\
\qquad\qquad \underline{-ab-bb}\\
\qquad\qquad\quad 0
\end{array}$$

Ex. 13.

$$\begin{array}{l}
3a-b)\;3a^3-12aa-baa+10ab-2bb\;(aa-4a+2b\\
\qquad\quad \underline{3a^3\qquad\quad -baa\qquad}\\
\qquad\qquad -12aa\qquad +10ab\\
\qquad\qquad -12aa\qquad \underline{+\;4ab\qquad\;}\\
\qquad\qquad\qquad\qquad\;\; +\;6ab-2bb\\
\qquad\qquad\qquad\qquad\;\; \underline{+\;6ab-2bb}\\
\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

3 RULE.

When the diviſor does not exactly divide the dividend; place the dividend over the diviſor, in form of a fraction; throwing out ſuch letters, as are found in all the terms of both the dividend and diviſor.

Ex. 14.

$$a-x)\ a\ \Big(\frac{a}{a-x} \text{ the quotient.}$$

Ex. 15.

$$ax-xx)\ ax+xx\Big(\frac{ax+xx}{ax-xx}=\frac{a+x}{a-x} \text{ quote.}$$

Ex. 16.

$$1-x)\,1\quad (1+x+xx+x^3+x^4+ \&\text{c. } \textit{ſine fine.}$$

$$\underline{1-x}$$

$$+x$$

$$\underline{x-xx} \qquad \left(\text{or } 1+x+xx+x^3+\frac{x^4}{1-x}.\right.$$

$$+xx$$

$$\underline{xx-x^3}$$

$$+x^3$$

$$\underline{x^3-x^4}$$

$$+x^4\ \&\text{c.}$$

Ex. 17.

$$aa-ee)\ aae \qquad \left(e+\frac{e^3}{aa}+\frac{e^5}{a^4}+\frac{e^7}{a^6-eea^4}\right.$$

$$\underline{aae-e^3}$$

$$+e^3$$

$$\underline{e^3-\frac{e^5}{aa}}$$

$$+\frac{e^5}{aa}$$

$$\underline{+\frac{e^5}{aa}-\frac{e^7}{a^4}}$$

$$+\frac{e^7}{a^4} \text{ Rem.}$$

This and ſuch like examples will be better underſtood after the next ſection.

Ex. 18.

$$a^3)\ 2a^5\ \left(\frac{2a^5}{a^3} = 2a^2.\right.$$

Ex. 19.

$$-2b^2)\ 18\,b^6\ \left(-\frac{18b^6}{b^2} = -9b^4.\right.$$

Ex. 20.

$$\overline{aa-xx}^{\,2})\ \overline{aa-xx}^{\,5}\ (\overline{aa-xx}^{\,3}$$
$$\underline{\overline{aa-xx}^{\,5}}$$
$$0$$

That like ſigns give +, and unlike ſigns —, in the quotient, will appear thus. The diviſor multiplied by the quotient muſt produce the dividend. Therefore, 1. When both are +, the quotient is +, becauſe then + × + muſt produce + in the dividend. 2. When they are both —, the quotient is + again, becauſe + × — muſt produce — in the dividend. Again, 3. When the diviſor is + and the dividend —, the quotient is —, becauſe — × + muſt produce — in the dividend. 4. Laſtly, If the diviſor is —, and the dividend +, the quotient will be —, becauſe — × — produces + in the dividend.

1. +)+(+
2. —)—(+
3. +)—(—
4. —)+(—

Cor. 1. *One power of a quantity, is divided by another power thereof; by ſubtracting the index of the diviſor, from the index of the dividend. Thus,* $\frac{a^5}{a^3} = a^{5-3} = a^2$. And $\frac{16b}{12b^3} = \frac{4b^{1-3}}{3} = \frac{4b^{-2}}{3} = \frac{4}{3bb}$.

Cor. 2. *Hence any power of a quantity may be taken out of the denominator and put into the numerator, and the contrary; by changing the ſign of the index.*

Thus $\frac{a}{2b^2} = \frac{ab^{-2}}{2}$. And $\frac{b}{a^{-3}} = ba^3$.

Cor. 3. *Hence* — *divided by* +, *or* + *divided by* —, *give the ſame quotient*, viz. —. *That is*, $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.

PROBLEM V.

To involve a quantity to any power.

1 RULE.

Multiply the quantity ſo often into itſelf as the index denotes. And where the root is +, all the powers are +. And where the root is —, all the odd powers are —, and all the even powers +.

Ex. 1.

a root	a^2 root
aa ſquare	a^4 ſquare
a^3 cube	a^6 cube
a^4 4th power	a^8 4th power
&c.	&c.

$-2a^3$ root
$+4a^6$ ſquare
$-8a^9$ cube
$+16a^{12}$ 4th power
&c.

Ex. 2.

ab root	$-3abb$ root
$aabb$ ſquare	$+9aab^4$ ſquare
a^3b^3 cube	$-27a^3b^6$ cube,
&c.	&c.

Ex. 3.

Involve $a+b$ *to the cube or 3d power.*

$$\begin{array}{lllll}
 & a+b & & & \\
 & a+b & & & \\
\hline
 & aa+ab & & & \\
 & \quad +ab+bb & & & \\
\hline
\text{ſquare} & aa+2ab+bb & & & \\
 & a+b & & & \\
\hline
 & a^3+2aab+abb & & & \\
 & \quad +aab+2abb+b^3 & & & \\
\hline
\text{cube} & a^3+3aab+3abb+b^3 & & & \\
\hline
\end{array}$$

2 RULE.

Multiply the index of the quantity, by the index of the power, and make the ſigns as in Rule 1.

Ex. 4.

root	a or a^1	$-2bba$; or $2b^2a$
ſquare	$a^{1\times2}$ or a^2	$+4b^{2\times2}a^{1\times2}$ or $+4b^4a^2$
cube	$a^{1\times3}$ or a^3	$-8b^{2\times3}a^{1\times3}$ or $-8b^6a^3$
m^{th} power	a^m	$-2^m\times b^{2m}a^m$

Ex. 5.

root	$\overline{a-x}^2$	
ſquare	$\overline{a-x}^{2\times2}$	or $\overline{a-x}^4$
cube	$\overline{a-x}^{2\times3}$	or $\overline{a-x}^6$
m^{th} power	$\overline{a-x}^{2\times m}$	or $\overline{a-x}^{2m}$

3 RULE.

3 RULE.

In a binomial. The power will conſiſt of 1 term more than the index of the power. The higheſt power of both is the index of the given power, and the index of the leading quantity continually decreaſes by one in every term, and in the following quantity, the indices of the terms are 0, 1, 2, 3, 4, *&c.*

Then for finding the unciæ or coefficients. The firſt is always 1; the ſecond, the index of the power. And in general, if the coefficient of any term be multiplied by the index of the leading quantity, and divided by the number of terms to that place; it gives the coefficient of the next following term.

Laſtly, When both terms of the root are +, all the terms of the power will be +; but if the ſecond term be —, then all the odd terms will be +, and all the even terms —.

Ex. 6.

Involve $a+e$ *to the 5th power.*

The ſeveral terms without the coefficients will be a^5, a^4e, a^3ee, a^2e^3, ae^4, e^5; and the coefficients $1, 5, \frac{5\times4}{2}, \frac{10\times3}{3}, \frac{10\times2}{4}, \frac{5\times1}{5}$;

that is, 1, 5, 10, 10, 5, 1.

And therefore the 5th power is

$$a^5+5a^4e+10a^3ee+10a^2e^3+5ae^4+e^5.$$

Ex. 7.

Involve $a-x$ *to the 4th power.*

the power is $a^4-4a^3x+\frac{4\times3}{2}a^2x^2-\frac{6\times2}{3}ax^3+\frac{4\times1}{4}x^4$;

that is, $a^4-4a^3x+6a^2x^2-4ax^3+x^4$.

4 RULE.

In trinomials, quadrinomials, *&c.* Let one letter remain, and put another letter for the reſt of the quantities; then involve this binomial by Rule 3; then inſtead of the powers of the aſſumed letter, find (by Rule 3.) the powers of the compound quantity it repreſents, which put in its ſtead.

Ex. 8.

Involve $a + b - x$ *to the third power.*

Put e for $b - x$, then the cube of $a + e$ is $a^3 + 3aae + 3aee + e^3$ (Rule 3), that is, $a^3 + 3aa \times \overline{b-x} + 3a \times \overline{b-x}^2 + \overline{b-x}^3$. But (Rule 3.) $\overline{b-x}^2 = bb - 2bx + xx$, and $\overline{b-x}^3 = b^3 - 3bbx + 3bx^2 - x^3$. Therefore $\overline{a+b-x}^3 = a^3 + 3aab - 3aax + 3abb - 6abx + 3axx + b^3 - 3bbx + 3bxx - x^3$.

Cor. 1. *The* n^{th} *power of* $a + e$, *that is,*

$$\overline{a+e}^n = a^n + na^{n-1}e + n \times \frac{n-1}{2} a^{n-2} ee + n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} e^3 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4} e^4 + \&c.$$

This rule is proved by involving $a + e$ as far as you will, for the ſeveral powers will always agree with the rule.

Cor. 2. *All powers of an affirmative quantity, are affirmative. And all odd powers of a negative quantity, are negative; and all even powers affirmative.*

Cor. 3. *The index of the power of any quantity, is the product of the index of the power, and index of the quantity.*

Cor. 4. *The* n^{th} *power of any product, is equal to the* n^{th} *power of each factor, multiplied together.*

$\overline{ab^2}|^{n} = a^n \times b^{2n}$.

PRO-

PROBLEM VI.

To extract the root of any quantity.

Evolution is juſt the reverſe to involution; and is performed as follows.

1 RULE.

For ſimple quantities; extract the root of the coefficient for the numerical part, and divide the index of the letter or letters, by the index of the power, gives the index of the root.

Ex. 1.

The cube root of a^3 *is* $a^{\frac{3}{3}}$ or a.

the ſquare root of $25a^4$ is $5a^{\frac{4}{2}}$ or $5aa$.

the ſquare root of $2a^2b^6$ is $a^{\frac{2}{2}}\,b^{\frac{6}{2}}\sqrt{2}$.
or $ab^3\sqrt{2}$.

the cube root of $-125b^9$ is $-5b^{\frac{9}{3}}$ or $-5b^3$.

2 RULE.

For the ſquare root, of a compound quantity; range the terms according to the dimenſions of ſome letter. Then find the root of the firſt term (1 Rule), and ſet it in the quotient: ſubtract its ſquare, and bring down the next term, which divide by double the quotient, and ſet the anſwer in the quotient. Multiply the diviſor and quotient by this laſt quotient, which ſubtract from the dividual. Proceed thus, juſt as in common arithmetic.

Ex. 2.

Extract the ſquare root of $aa+4ab+4bb-2ax-4bx+xx$.

$$\begin{array}{rl}
 & aa+4ab+4bb-2ax-4bx+xx\ (a+2b-x \text{ root} \\
 & aa \\ \hline
2a+2b) & 4ab+4bb \\
 & 4ab+4bb \\ \hline
 & 0 \\
2a+4b-x) & -2ax-4bx+xx \\
 & -2ax-4bx+xx \\ \hline
\end{array}$$

Ex. 3.

Extract the ſquare root of $aa-6na+2za+9nn-6nz+zz$.

$$\begin{array}{rlll}
 & aa\begin{matrix}-6n\\+2z\end{matrix}a & \begin{matrix}+9nn\\-6nz\end{matrix} & \left(a\begin{matrix}-3n\\+z\end{matrix}\right. \\
 & aa & +zz & \\ \hline
 & & +9nn & \\
\left.2a\begin{matrix}-3n\\+z\end{matrix}\right) & 0\begin{matrix}-6n\\+2z\end{matrix}a & \begin{matrix}-6nz\\+zz\end{matrix} & \\
 & \begin{matrix}-6n\\+2z\end{matrix}a & \begin{matrix}+9nn\\-6nz\\+zz\end{matrix} & \\ \hline
 & 0 & 0 &
\end{array}$$

Ex. 4.

Extract the ſquare root of $aa+xx$.

$$aa+xx\left(a+\frac{xx}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\&c.\right.$$

$$aa$$

$$\left.2a+\frac{xx}{aa}\right)\ 0+xx$$

$$+xx+\frac{x^4}{4aa}$$

$$\left.2a+\frac{xx}{a}-\frac{x^4}{8a^3}\right)\ 0-\frac{x^4}{4aa}$$

$$-\frac{x^4}{4aa}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}$$

$$\left.2a+\right)\quad 0+\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$$

3 RULE.

In higher powers. Find the root of the firſt member, which place in the quotient: ſubtract its power, the remainder is the reſidual. Involve this root to the next lower power, and multiply it by the index of the given power, for a diviſor; by this divide the firſt term of the reſidual, the quotient is the next term of the root. Then involve the whole root as before, and ſubtract: and repeat the operation, till all the terms of the root be had.

Ex. 5.

Extract the cube root of $x^6+6x^5-40x^3+96x-64$.

$$x^6+6x^5-40x^3+96x-64\ (xx+2x-4 \text{ root.}$$

$$x^6$$

$$3x^4)\ 6x^5\ (+2x$$

$$x^6+6x^5+12x^4+8x^3=\overline{xx+2x}^3.$$

$$3x^4)\ 0-12x^4(-4$$

$$x^6+6x^5-40x^3+96x-64=\overline{xx+2x-4}^3.$$

$$0$$

Ex. 6.

Extract the 4th root of $16a^4-96a^3b+216aabb-216ab^3+81b^4$.

$$\begin{array}{lr}
16a^4-96a^3b+216aabb-216ab^3+81b^4\,(2a-4b & \\
16a^4 & \text{root.} \\
\hline
\end{array}$$

$$\begin{array}{l}
24a^3)\ 0\ -96a^3b\ (-4b \\
\qquad\quad -96a^3b \\
\hline
16a^4-96a^3b+216aabb-216ab^3+81b^4 \\
\hline
\qquad 0 \\
\hline
\end{array}$$

4 RULE.

The roots of compound quantities, may ſometimes be diſcovered thus. Extract the roots out of all the ſimple powers or terms in it; then connect theſe roots by the ſigns + or —, as you judge will beſt anſwer. Involve this compound root to the proper power; then if it be the ſame with the given quantity, you have got the root. If it only differs in the ſigns, change ſome of them, till its power agrees with the given one throughout.

Ex. 7.

To extract the cube root of $a^3-6a^2b+12ab^2-8b^3$.

Here the root of a^3 is a, and the root of $-8b^3$ is $-2b$. Then $a-2b$ is the root, for its cube is $a^3-6a^2b+12ab^2-8b^3$, as required.

Ex. 8.

Extract the 4th root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$.

The roots of $16a^4$ and $81x^4$, are $2a$ and $3x$. Therefore if $2a+3x$ be made the root and involved,

volved, it is $16a^4+96a^3x+216aaxx+216ax^3+81x^4$, which differs in the ſigns, from the quantity given. Therefore make $2a-3x$ the root, which being involved ſucceeds; the power being $16a^4-96a^3x+216aaxx-216ax^3+81x^4$.

5 RULE.

When the quantity given has not ſuch a root as is required, ſet it down in form of a ſurd.

Ex. 9.

Square root of a^3, is $\sqrt{a^3}$.
Cube root of $15aa$, is $\sqrt[3]{15aa}$.
4th root of $2a^5x^3$, is $\sqrt[4]{2a^5x^3}$.

Ex. 10.

The cube root of $a^3-6a^2b+12abb+8b^3$, is
$\sqrt[3]{a^3-6a^2b+12abb+8b^3}$.

Ex. 11.

What is the 5th root of a^5-x^5.
the root is $\sqrt[5]{a^5-x^5}$.

Cor. 1. *The ſquare root, or any even root, of an affirmative quantity, may be either + or —.*

For the ſquare root of aa may be $+a$ or $-a$, for $+a\times+a=aa$, and $-a\times-a=aa$: alſo the 4th root of a^4 is $+a$ or $-a$, for the 4th power of $-a$ is $+a^4$, as well as of $+a$.

Cor. 2. *Any odd root of a quantity, will have the ſame ſign, as the quantity itſelf.*

For the root of $+a^3$ and $-a^3$, will be $+a$ and $-a$; for $+a$ cubed is $+a^3$, and $-a$ cubed is $-a^3$.

Cor. 3. *The ſquare root, or any even root, of a negative quantity, is impoſſible.*

For neither $+a \times +a$, nor $-a \times -a$, can produce $-aa$.

Cor. 4. *The* n^{th} *root of a product, is equal to the* n^{th} *root of each of the factors, multiplied together:*

$$\sqrt[n]{AB} = \sqrt[n]{A} \times \sqrt[n]{B}.$$

* * *

SECT. II.

Of FRACTIONS.

THE operations of algebraic fractions are exactly the ſame as thoſe of vulgar fractions in arithmetic; therefore he that has made himſelf maſter of vulgar fractions, will eaſily underſtand how to manage all ſorts of algebraic fractions, as in the following problems.

PROBLEM VII.

To reduce a given quantity to a fraction of a given denominator.

RULE.

Multiply that quantity by the given denominator, and under the product write the ſame denominator.

Ex. 1.

Let $a+b$ *have the denominator* x.

$$\frac{\overline{a+b}\times x}{x} = \frac{ax+bx}{x}, \text{ anſwer.}$$

Ex. 2

Let $xx-yy$ *have the denominator* 1.

$$\frac{\overline{xx-yy}\times 1}{1} = \frac{xx-yy}{1}, \text{ anſwer.}$$

Ex. 3.

Let $\frac{a}{b}$ *have the denominator* $b-c$.

$$\frac{\frac{a}{b}\times\overline{b-c}}{b-c} = \frac{a-\frac{ac}{b}}{b-c}, \text{ anſwer.}$$

Cor *The value of a fraction is not altered, by multiplying both numerator and denominator by the ſame quantity. Thus* $\frac{rab}{rc} = \frac{rabd}{rcd} = \frac{ab}{c}$.

PROBLEM VIII.

To reduce a mixed number to a fraction.

RULE.

Multiply the integral part by the denominator of the fraction, and to the product add the numerator, under which write the common denominator.

Ex. 1.

Let $a - \frac{b}{c}$ *be given.* Then $\frac{ac-b}{c}$ is the fraction required.

Ex. 2.

Suppoſe $a-x+\frac{aa-ax}{x}$.

Here $\frac{ax-xx+aa-ax}{x}$, or $\frac{aa-xx}{x}$ is that required.

PROBLEM IX.

To reduce an improper fraction to a whole or mixed number.

RULE.

Divide the numerator by the denominator, as far as you can, gives the integral part; and place the remainder over the denominator for the fractional part.

Ex. 1.

Given $\frac{ab-aa}{b}$.

$$\begin{array}{l} b)\;ab-aa\;\left(\;a-\dfrac{aa}{b}\right.\text{ anſwer.} \\ \underline{ab} \\ -aa \end{array}$$

Ex. 2.

Suppoſe $\frac{aa+xx}{a-x}$.

$$\begin{array}{l} a-x)\;aa+xx\;\left(\;a+x+\dfrac{2xx}{a-x}\right.\text{, anſwer,} \\ \underline{aa-ax} \\ +ax+xx \\ \underline{+ax-xx} \\ +2xx \end{array}$$

PROBLEM X.

To find the greateſt common diviſor, for the terms of a fraction, or for any two quantities.

RULE.

The quantities being ranged according to the dimenſions of ſome letter; divide the greater by the leſſer, and the laſt diviſor by the laſt remainder, and ſo on continually till nothing remain; then the laſt diviſor is that required. But obſerve, firſt to throw out of each diviſor, all the ſimple diviſors, (or others) that will divide it; and then proceed. The ſimple diviſors are had by inſpection.

Ex. 1.

Let $\frac{cd+dd}{aac+aad}$ *be the fraction propoſed.*

$$\begin{array}{ll} & cd+dd)\;aac+aad\;(\\ \text{or} & c+d\;)\;aac+aad\;(aa \\ & \underline{aac+aad} \\ & 0 \end{array}$$

Therefore $c+d$ is the greateſt common diviſor.

$$c+d\Big)\frac{cd+dd}{aac+aad}=\frac{d}{aa}.$$

Ex. 2.

Let $\dfrac{a^3-abb}{aa+2ab+bb}$ *be propoſed.*

$$\begin{array}{rll}
aa+2ab+bb) & a^3-abb & (a \\
 & \underline{a^3+2aab+abb} & \\
 & \underline{-2aab-2abb} & \text{remainder.} \\
 & & \\
-2aab-2abb) & aa+2ab+bb & (\\
\text{or } a+b\quad) & aa+2ab+bb & (a+b \\
 & \underline{aa+ab} & \\
 & \quad +ab+bb & \\
 & \quad \underline{+ab+bb} & \\
 & \qquad 0 &
\end{array}$$

Therefore $a+b$ is the greateſt common diviſor.

Ex. 3.

Suppoſe $\dfrac{a^4-b^4}{a^5-bba^3}$ *be given.*

$$\begin{array}{rll}
a^4-b^4) & a^5-bba^3 & (a \\
 & \underline{a^5-b^4a} & \\
\text{rem. } -bba^3+b^4a) & a^4-b^4 & (\\
\text{or}\quad aa-bb) & a^4-b^4 & (aa+bb \\
 & \underline{a^4-bbaa} & \\
 & \quad +bbaa-b^4 & \\
 & \quad \underline{+bbaa-b^4} & \\
 & \qquad 0 &
\end{array}$$

the common diviſor is $aa-bb$.

PRO-

PROBLEM XI.

To reduce a fraction to its loweſt terms.

RULE.

Find the greateſt common meaſure (Prob. X), by which divide both numerator and denominator of the fraction; the quotients will be the numerator and denominator of the fraction required.

Ex. 1.

Let $\frac{cc+dd}{aac+aad}$ *be propoſed.*

the greateſt common diviſor is $c+d$. Therefore

$$c+d\Big)\frac{cd+dd}{aac+aad}=\frac{d}{aa}$$ the fraction required.

Ex. 2.

Let $\frac{a^3-abb}{aa+2ab+bb}$ *be propoſed.*

Here $a+b$ is the greateſt common diviſor: then

$$a+b\Big)\frac{a^3-abb}{aa+2ab+bb}=\frac{aa-ab}{a+b}$$ the fraction ſought.

Ex. 3.

Suppoſe $\frac{a^4-b^4}{a^5-bba^3}$ *to be given.*

the greateſt common diviſor is $aa-bb$; then

$$aa-bb\Big)\frac{a^4-b^4}{a^5-bba^3}\Big(=\frac{aa+bb}{a^3}$$ the fraction required.

PROBLEM XII.

To reduce fractions of different denominators, to fractions of the ſame value, having a common denominator

1 RULE.

Multiply each numerator, into all the other denominators, for a new numerator; then multiply all the denominators together for a common denominator.

Ex. 1.

Let $\frac{a}{b}$, $\frac{a+b}{c}$ *be given.*

theſe become $\frac{ac}{bc}$, $\frac{ab+bb}{bc}$.

Ex. 2.

Let $\frac{a}{b}$, $\frac{c}{d}$, $\frac{f}{g}$ *be propoſed.*

they become $\frac{adg}{bdg}$, $\frac{cbg}{bdg}$, $\frac{fbd}{bdg}$.

2 RULE.

Divide the denominators by their greateſt common diviſor, then multiply both numerator and denominator of each fraction, by all the other quotients, which will produce as many new fractions.

Ex. 3.

Suppoſe $\frac{a}{2bb}$, $\frac{c}{2b}$, $\frac{d}{b}$.

$$2b \quad 2 \quad 1$$

$\frac{2a}{4bb}$, $\frac{2bc}{4bb}$, $\frac{4bd}{4bb}$ the fractions required.

or $\frac{a}{2bb}$, $\frac{bc}{2bb}$, $\frac{2bd}{2bb}$.

Ex. 4.

$$\text{Given } \frac{2aa}{aa-ab},\ \frac{3ab-2bb}{2ac}.$$

$$a-b \qquad\qquad 2c$$

$$\frac{4aac}{2aac-2abc},\ \frac{\overline{3ab-2bb}\times\overline{a-b}}{2aac-2abc} \quad \text{that is,}$$

$$\frac{4aac}{2aac-2abc},\ \frac{3aab-5abb+2b^3}{2aac-2abc}.$$

PROBLEM XIII.

To add fractional quantities together.

RULE.

If the fractions have not a common denominator, reduce them to one (Prob. XII); then add the numerators, and under the ſum, write the common denominator.

Ex. 1.

$$\text{Add } \frac{a}{b} \text{ to } \frac{c}{d}$$

reduced $\frac{ad}{bd}$ and $\frac{bc}{bd}$; then $\frac{ad+bc}{bd}$ = ſum.

Ex. 2.

$$\text{Add } \frac{a}{b}, \frac{c}{d}, \frac{f}{g} \text{ together.}$$

reduced $\frac{adg}{bdg}$, $\frac{bcg}{bdg}$, $\frac{bdf}{bdg}$; then $\frac{adg+bcg+bdf}{bdg}$ = ſum.

Ex. 3.

$$\text{Add } \frac{a-b}{3c} \text{ to } \frac{2b-4a+d}{3c},$$

the ſum $= \frac{a-b+2b-4a+d}{3c} = \frac{b-3a+d}{3c}.$

Ex. 4.

$$\text{To } a - \frac{aa}{b}$$

$$\text{add } b + \frac{a-b}{c}$$

$$\text{ſum } a + b + \frac{ab - bb - caa}{cb}.$$

PROBLEM XIV.

To ſubtract one fraction from another.

RULE.

Reduce them to a common denominator; then ſubtract the numerators: and under the difference, write the common denominator.

Ex. 1.

$$\text{From } \frac{a+b}{d} \text{ ſubtract } \frac{c}{d}.$$

$$\frac{a+b-c}{d} = \text{difference.}$$

Ex. 2.

$$\text{rom } \frac{a+b}{d} = \frac{ab+bb}{bd}$$

$$\text{ſubtract } \frac{aa}{b} = \frac{aad}{bd};$$

$$\text{then } \frac{ab+bb-aad}{bd} = \text{remainder.}$$

Ex. 3.

$$\text{From } \frac{a-b}{3c} \text{ take } \frac{2b-4a}{5d}.$$

$$\text{reduced } \frac{5ad-5bd}{15cd}, \frac{6bc-12ac}{15cd}.$$

$$\text{remainder } \frac{5ad-5bd-6bc+12ac}{15cd}.$$

Ex. 4.

$$\text{From } a - \frac{aa}{b}, \text{ or } a - \frac{aac}{bc}$$

$$\text{take } b + \frac{a-b}{c}, \text{ or } b + \frac{ab-bb}{bc}$$

$$\text{difference } a-b + \frac{-aac-ab+bb}{bc}$$

PROBLEM XV.

To multiply fractions.

1 RULE.

In fractions, multiply the numerators together for a new numerator; and multiply the denominators together for a new denominator.

Ex. 1.

$$\text{Multiply } \frac{a}{b} \text{ by } \frac{c}{d}.$$

$$\text{then } \frac{a \times c}{b \times d} \text{ or } \frac{ac}{bd} = \text{product.}$$

Ex. 2.

$$\text{Multiply } \frac{b}{c} \text{ by } \frac{a+b}{b+c}.$$

$$\text{here } \frac{b}{c} \times \frac{a+b}{b+c} = \frac{ab+bb}{bc+cc}, \text{ product.}$$

Ex. 3.

$$\text{Multiply } \frac{aa-bb}{bc} \text{ by } \frac{aa+bb}{b+c}.$$

$$\text{then } \frac{aa-bb}{bc} \times \frac{aa+bb}{b+c} = \frac{a^4-b^4}{bbc+bcc}, \text{ product.}$$

2 RULE.

When the numerator of one, and denominator of the other, can be divided by ſome common diviſor, take the quotients inſtead thereof.

Ex. 4.

$$\text{Let } \frac{c}{aa} \text{ multiply } \frac{aabb}{3cdd}.$$

$$\text{reduced } \frac{1}{1} \times \frac{bb}{3dd} = \frac{bb}{3dd}, \text{ product.}$$

Ex. 5.

$$\text{Multiply } \frac{aa+2ab+bb}{cd-dd} \text{ by } \frac{dd}{a+b}.$$

$$\text{reduced } \frac{a+b}{c-d} \times \frac{d}{1} = \frac{ad+bd}{c-d}, \text{ product.}$$

3 RULE.

If a fraction is to be multiplied by an integer, which happens to be the ſame with the denominator; take the numerator for the product.

Ex. 6.

$$\text{Multiply } \frac{aa-2bb}{a-b} \text{ by } a-b.$$

$$\text{quotient } aa-2bb.$$

4 RULE.

When a fraction is to be multiplied by an integer; multiply the numerator by the integer.

Ex. 7.

$$\text{Multiply } \frac{aa+3bb}{3cd} \text{ by } xx.$$

$$\text{then } \frac{aaxx+3bbxx}{3cd} \text{ or } \frac{aa+3bb}{3cd}\,xx = \text{the prod.}$$

Ex. 8.

Multiply $\frac{2a - 2x}{3b}$ by $a + x$

then $\frac{2aa - 2xx}{3b}$ = product.

Ex. 9.

Multiply $a + \frac{b - c}{d}$.

by $b - \frac{b + c}{d}$

product $ab + \frac{bb - bc}{d} - \frac{ab + ac}{d} - \frac{bb - cc}{dd}$.

Schol. *By this rule, a compound fraction may be reduced to a ſimple one.*

PROBLEM XVI.

To divide one fraction by another.

1 RULE.

In fractions, multiply the denominator of the diviſor by the numerator of the dividend, for a new numerator; alſo multiply the numerator of the diviſor into the denominator of the dividend, for a new denominator.

Ex. 1.

Divide $\frac{a}{b}$ *by* $\frac{c}{d}$.

$\frac{c}{d}\Big)\frac{a}{b}\Big(\frac{ad}{bc}$ the quotient.

Ex. 2.

$$\text{Let } \frac{a+c}{a-b} \text{ divide } \frac{a+b}{a}.$$

$$\frac{a+c}{a-b}\Big)\ \frac{a+b}{a}\ \Big(\frac{aa-bb}{aa+ac} \text{ quotient.}$$

2 RULE.

If the fractions have a common denominator; take the numerator of the dividend, for a numerator; and the numerator of the diviſor, for the denominator.

Ex. 3.

$$\text{Divide } \frac{aa-bb}{a+d} \text{ by } \frac{2ab-bb}{a+d}:$$

$$\text{quotient } \frac{aa-bb}{2ab-bb}.$$

Ex. 4.

$$\text{Let } \frac{aa+2ab+bb}{c-d} \text{ divide } \frac{a^3-abb}{c-d},$$

$$\text{then } \frac{a^3-abb}{aa+2ab+bb} = \text{quotient.}$$

$$\text{or } \frac{aa-ab}{a+b} = \text{quotient reduced.}$$

3 RULE.

When fractions are to be divided by integers; multiply the denominators of the fractions, by ſuch integers.

Ex. 5.

$$\text{Divide } \frac{a-b}{c} \text{ by } d.$$

$$\text{quotient is } \frac{a-b}{cd}.$$

Ex. 6.

Let $a+b$ divide $\frac{aa-2bb}{a-b}$.

then $\frac{aa-2bb}{\overline{a-b}\times\overline{a+b}} = \frac{aa-2bb}{aa-bb}$, quotient.

4 RULE.

When the two numerators, or the two denominators, can be divided by ſome common diviſor; throw out ſuch diviſor, and proceed by Rule 1.

Ex. 7.

Let $\frac{a-b}{cd}$ *divide* $\frac{aa-bb}{c+d}$,

reduced $\frac{1}{cd}\Big)\frac{a+b}{c+d}\Big(\frac{acd+bcd}{c+d}$, quotient.

Ex. 8.

Let $\frac{aa+ab}{a-b}$ *divide* $\frac{a^4-b^4}{aa-2ab+bb}$:

reduced $\frac{a}{1}\Big)\frac{a^3-a^2b+abb-b^3}{a-b}\Big(\frac{a^3-aab+abb-b^3}{aa-ab}$ the quotient.

that is, the quotient $= a+\frac{bb}{a}$.

From hence may be deduced the following corollaries.

Cor. 1. *The value of any fractional quantity is not at all changed, by changing all the ſigns of both numerator and denominator. Thus* $\frac{ab-ac}{r-c} = \frac{ac-ab}{c-r}$.

Cor. 2. *The value of any compound fractional quantity, is equal to the ſum of all the particular ſimple*

ſimple fractions, that compoſe it. Thus

$$\frac{rx+2cx-11rz}{3r-2x}=\frac{rx}{3r-2x}+\frac{2cx}{3r-2x}-\frac{11rz}{3r-2x}.$$

Cor. 3. *If a fraction be multiplied by any given quantity; it is the ſame thing whether the numerator be multiplied by that quantity, or the denominator divided by it.* $\frac{dab}{dc}\times d=\frac{dabd}{dc}=\frac{dab}{c}$.

Cor. 4. *The product of two fractions, is equal to the fraction, that has the product of the numerators for the numerator; and the product of the denominators for its denominator.*

$$\frac{a}{b+x}\times\frac{r-c}{x}=\frac{a\times\overline{r-c}}{\overline{b+x}\times x}=\frac{ar-ac}{bx+xx}.$$

Cor. 5. *If a fraction is to be divided by ſome quantity; it is the ſame thing whether the numerator be divided by it, or the denominator multiplied.*

For $\frac{2az}{x}\div r=\frac{2az}{rx}$. And $\frac{2ar}{x}\div r=\frac{2a}{x}$.

Cor. 6. *If any ſort of quantity is to be divided by a fraction; it is the ſame thing, as to multiply the ſaid quantity, by the fraction inverted. Thus*

$ab\div\frac{r}{s}=ab\times\frac{s}{r}$. *And* $\frac{a}{c}\div\frac{b}{r}$ or

$$\frac{\frac{a}{c}}{\frac{b}{r}}=\frac{a}{c}\times\frac{r}{b}=\frac{ar}{bc}.$$

PROBLEM XVII.

To involve fractional quantities.

RULE.

Involve the numerator into itſelf, for a new numerator; and the denominator into itſelf for a new denominator; each as often as the index of the power.

Ex.

Ex. 1.

Involve $\frac{a}{b}$ *and* $\frac{1}{aa}$.

root	$\frac{a}{b}$	$\frac{1}{aa}$
ſquare	$\frac{aa}{bb}$	$\frac{1}{a^4}$
cube	$\frac{a^3}{b^3}$	$\frac{1}{a^6}$
4th power	$\frac{a^4}{b^4}$	$\frac{1}{a^8}$
&c.		

Ex. 2.

Let $\frac{3bc}{2ad}$, *and* $\frac{-ad}{4bb}$, *be involved.*

root	$\frac{3bc}{2ad}$	$-\frac{ad}{4bb}$
ſquare	$\frac{9bbcc}{4aadd}$	$+\frac{aadd}{16b^4}$
cube	$\frac{27b^3c^3}{8a^3d^3}$	$-\frac{a^3d^3}{64b^6}$
&c.		

Ex. 3.

Involve $\frac{aa-bc}{a+c}$ *to the ſquare,* &c.

$\frac{a^4-2aabc+bbcc}{aa+2ac+cc}$, *the ſquare.*

$\frac{a^5-3a^4bc+3a^2bbcc-b^3c^3}{a^3+3a^2c+3ac^2+c^3}$, cube, &c.

Ex. 4.

Involve $\frac{a-x}{2b}$ *to the 4th power.*

it is $\frac{a^4-4a^3x+6aaxx-4ax^3+x^4}{16b^4}$.

or thus $\left.\overline{\frac{a-x}{2b}}\right|^4$ or $\frac{\overline{a-x}^4}{16b^4}$

PROBLEM XVIII.

To extract the root of a fraction.

RULE.

Extract the proper root of both numerator and denominator, if it can be done. If not ſet the radical ſign ($\surd$) before one or both of them, as they happen to be ſurd.

Ex. 1.

What is the ſquare root of $\frac{9a^2b^4}{4dd}$.

root $\frac{3abb}{2d}$.

Ex. 2.

What the cube root of $\frac{a^3-3a^2b+3ab^2-b^3}{a^3+3a^2b+3ab^2-b^3}$.

the root is $\frac{a-b}{a+b}$.

Ex. 3.

The ſquare root of $\frac{aabb}{d^3}$, is $\sqrt{\frac{aabb}{d^3}}$ or $\frac{ab}{\sqrt{d^3}}$.

Ex. 4.

What is the cube root of $\frac{-27a^3b^3}{a^3-b^3}$.

the root is $\frac{-3ab}{\sqrt{a^3-b^3}}$.

Ex. 5.

What is the cube root of $\frac{a^3+4abd-d^3}{8a^6b^3}$.

the root is $\sqrt[3]{\frac{a^3+4abd-d^3}{8a^6b^3}}$, or $\frac{\sqrt{a^3+4abd-d^3}}{2aab}$.

Ex. 6.

What is the 4th root of $\frac{x^4-y^4}{8ax^3-8x^2yy+y^4}$.

the root is $\sqrt[4]{\frac{x^4-y^4}{8ax^3-8x^2yy+y^4}}$,

or $\frac{\sqrt{x^4-y^4}}{\sqrt[4]{8ax^3-8x^2y^2+y^4}}$.

Cor. *The n^{th} power or root of a fraction, is equal to the n^{th} power or root of the numerator, divided by the n^{th} power or root of the denominator.*

$\overline{\frac{a}{x}}\Big|^{n} = \frac{a^n}{x^n}$. And $\sqrt[n]{\frac{a}{x}} = \frac{\sqrt[n]{a}}{\sqrt[n]{x}}$.

SECT. III.

Of SURDS.

SURDS are ſuch quantities as have not a proper root. *Simple Surds* are thoſe which conſiſt but of one term. *Compound Surds* are thoſe which conſiſt of ſeveral ſimple ones. And *Univerſal Surds* are thoſe conſiſting of ſeveral terms under any radical ſign.

Surds are ſaid to be *commenſurable*, when they are as one number to another; and *incommenſurable*, when their proportion cannot be expreſſed in numbers.

PROBLEM XIX.

To deſignate or expreſs the roots of quantities by fractional indices.

1 RULE.

Divide the index of the quantity by the number expreſſing the root; the quotient is the index of the root required.

Ex. 1.

Let the quantity a be propoſed.

then $\sqrt{a} = a^{\frac{1}{2}}$, $\sqrt[3]{a} = a^{\frac{1}{3}}$, $\sqrt[4]{a} = a^{\frac{1}{4}}$, &c.

Ex. 2.

Let $3ab^2$ *be propoſed.*

$$\sqrt{3ab^2} = \overline{3abb}|^{\frac{1}{2}} = a^{\frac{1}{2}} b\sqrt{3}. \quad \sqrt[3]{3ab^2} = a^{\frac{1}{3}} b^{\frac{2}{3}} \sqrt[3]{3}.$$

$$\sqrt[4]{3ab^2} = a^{\frac{1}{4}} b^{\frac{1}{2}} \sqrt[4]{3}. \quad \sqrt[5]{3abb} = a^{\frac{1}{5}} b^{\frac{2}{5}} \sqrt{3}, \text{ \&c.}$$

Ex. 3.

Let a^3 *be given.*

then $\sqrt{a^3} = a^{\frac{3}{2}}$, $\sqrt[3]{a^3} = a^{\frac{3}{3}}$ or a, $\sqrt[4]{a^3} = a^{\frac{3}{4}}$, &c.

Ex. 4.

Let $\overline{aa - xx}$ *be propoſed.*

then $\sqrt{aa - xx} = \overline{aa - xx}^{\frac{1}{2}}$,

$\sqrt[3]{aa - xx} = \overline{aa - xx}^{\frac{1}{3}}$, &c.

Ex. 5.

Let $\frac{1}{x}$ *be given.*

then $\sqrt{\frac{1}{x}} = \frac{1}{x^{\frac{1}{2}}}$. $\sqrt[3]{\frac{1}{x}} = \frac{1}{x^{\frac{1}{3}}}$, $\sqrt[4]{\frac{1}{x}} = \frac{1}{x^{\frac{1}{4}}}$ &c.

Ex. 6.

Let $\frac{a - b}{abbc^3}$ *be given.*

then $\sqrt{\frac{a-b}{abbc^3}} = \frac{\overline{a-b}^{\frac{1}{2}}}{a^{\frac{1}{2}}bc^{\frac{3}{2}}}$. $\sqrt[3]{\frac{a-b}{abbc^3}} = \frac{\overline{a-b}^{\frac{1}{3}}}{a^{\frac{1}{3}}b^{\frac{2}{3}}c}$, &c.

Ex. 7.

Let $\frac{aa}{b^3}$ *be propoſed.*

$\sqrt{\frac{aa}{b^3}} = \frac{a}{b^{\frac{1}{2}}}$, $\sqrt[3]{\frac{aa}{b^3}} = \frac{a^{\frac{2}{3}}}{b}$, $\sqrt[4]{\frac{aa}{b^3}}$, $= \frac{a^{\frac{1}{2}}}{b^{\frac{3}{4}}}$, &c.

Ex. 8.

Let $\frac{a+2x}{aa-xx}$ *be propoſed.*

$$\sqrt{\frac{a+2x}{aa-xx}} = \frac{\overline{a+2x}|^{\frac{1}{2}}}{\overline{aa-xx}|^{\frac{1}{2}}}, \quad \sqrt[3]{\frac{a+2x}{aa-xx}} = \frac{\overline{a+2x}|^{\frac{1}{3}}}{\overline{aa-xx}|^{\frac{1}{3}}}$$

$$= \overline{\frac{a+2x}{aa-xx}}\Big|^{\frac{1}{3}}.$$

2 RULE.

When any quantity is in the denominator of a fraction; ſet it in the numerator, and change the ſign of the index.

Ex. 9.

$$\sqrt{\frac{1}{a}} = \frac{1}{a^{\frac{1}{2}}} = \frac{a^{-\frac{1}{2}}}{1} = a^{-\frac{1}{2}}.$$

Ex. 10.

Let $\frac{1}{a}$, $\frac{1}{aa}$, $\frac{1}{a^3}$, $\frac{1}{a^4}$, $\frac{1}{a^5}$, &c. *be given.* then they become a^{-1}, a^{-2}, a^{-3}, a^{-4}, a^{-5}, &c. reſpectively.

Ex. 11.

Given $\frac{ab}{x^2y^3}$. This becomes $abx^{-2}y^{-3}$.

Ex. 12.

Given $\frac{1}{aa-xx}$, $\frac{1}{\overline{aa-xx}|^2}$, $\frac{1}{\overline{aa-xx}|^3}$, &c. they become $\overline{aa-xx}|^{-1}$, $\overline{aa-xx}|^{-2}$, $\overline{aa-xx}|^{-3}$, &c.

Ex. 13.

Let $\frac{aab}{a+x}$, $\frac{aab}{\overline{a+x}^{2}}$, $\frac{aab}{\overline{a+x}^{3}}$ *be given.*

they are $aab \times \overline{a+x}^{-1}$, $aab \times \overline{a+x}^{-2}$, $aab \times \overline{a+x}^{-3}$.

In order to explain this; let there be a rank of powers; as 1, a, aa, aaa, $aaaa$, $aaaaa$, &c. the ſame will (by Def. 20.) be denoted by 1, a, a^2, a^3, a^4, a^5, &c. Now theſe quantities, a, a^2, a^3, &c. are in geometrical progeſſion, and their indices, in arithmetic progreſſion; as is plain. Now ſince 1, a, a^2, a^3, &c. are geometrical proportionals, therefore theſe will alſo be geometrical proportionals,

1, $\sqrt{a}$, $\sqrt{a^2}$, $\sqrt{a^3}$, $\sqrt{a^4}$, $\sqrt{a^5}$, $\sqrt{a^6}$, $\sqrt{a^7}$, $\sqrt{a^8}$, &c.

that is 1, $\sqrt{a}$, a, $\sqrt{a^3}$; aa, $\sqrt{a^5}$, a^3, $\sqrt{a^7}$, a^4, &c.

and 1, $\sqrt[3]{a}$, $\sqrt[3]{aa}$; a, $\sqrt[3]{a^4}$, $\sqrt[3]{a^5}$, a^2, $\sqrt[3]{a^7}$, $\sqrt[3]{a^8}$, &c.

and 1, $\sqrt[4]{a}$, $\sqrt[4]{aa}$, $\sqrt[4]{a^3}$, a, $\sqrt[4]{a^5}$, $\sqrt[4]{a^6}$, $\sqrt[4]{a^7}$, aa, &c. and ſo on.

Therefore by the rule of analogy, the indices of all theſe, are alſo in arithmetic progreſſion.

Take any one of theſe ſeries as 1, $\sqrt[3]{a}$, $\sqrt{aa}$, a, $\sqrt[3]{a^4}$, &c. theſe will be equivalent to 1, $a^{\frac{1}{3}}$ $a^{\frac{2}{3}}$, a, $a^{\frac{4}{3}}$, &c.

Suppoſe now the ſeries 1, a, a^2, a^3, &c. continued backwards, the powers of a will come into the denominator; and the indices, which continually decreaſe, will then become negative, and will ſtand thus:

Powers $\frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a}, \ldots 1, a, a^2, a^3, a^4$, &c.

Indices $-3, -2, -1, 0, 1, 2, 3, 4$, &c.

therefore $a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, a^4$, &c. will repreſent theſe powers; that is,

$$\frac{1}{a^3}=a^{-3},\ \frac{1}{a^2}=a^{-2},\ \frac{1}{a}=a^{-1},\ 1=a^0,\ \&c.$$

In like manner, let the ſeries 1, $a^{\frac{1}{3}}, a^{\frac{2}{3}}, a^1, a^{\frac{4}{3}}$, &c. be continued backwards; theſe powers, and their indices will be as follows:

$$\frac{1}{a^{\frac{4}{3}}},\ \frac{1}{a},\ \frac{1}{a^{\frac{2}{3}}},\ \frac{1}{a^{\frac{1}{3}}},\ 1,\ a^{\frac{1}{3}},\ a^{\frac{2}{3}},\ a^1,\ a^{4},\ \&c.$$

$$-\frac{4}{3},\ -1,\ -\frac{2}{3},\ -\frac{1}{3},\ 0,\ \frac{1}{3}\ \ \frac{2}{3},\ 1,\ \frac{4}{3},\ \&c.$$

Then,

$$a^{-\frac{4}{3}},\ a^{-1},\ a^{-\frac{2}{3}},\ a^{-\frac{1}{3}},\ a^0,\ a^{\frac{1}{3}},\ a^{\frac{2}{3}}, a, a^{\frac{4}{3}},\ \&c.$$

will denote the ſame powers; that is,

$$\frac{1}{a^{\frac{4}{3}}}=a^{-\frac{4}{3}},\ \frac{1}{a}=a^{-1},\ \frac{1}{a^{\frac{2}{3}}}=a^{-\frac{2}{3}},\ \frac{1}{a^{\frac{1}{3}}}=a^{-\frac{1}{3}},$$

$a^0=1$, &c. And therefore the ſeries,

$$\frac{1}{\sqrt[3]{aaaa}},\ \frac{1}{\sqrt[3]{aaa}},\ \frac{1}{\sqrt[3]{aa}},\ \frac{1}{\sqrt[3]{a}},\ 1,\ \sqrt[3]{a},$$

$\sqrt[3]{aa}, \sqrt[3]{aaa}, \sqrt[3]{aaaa}$, &c. may be expreſſed thus,

$$\frac{1}{\sqrt[3]{a^4}},\ \frac{1}{\sqrt[3]{a^3}},\ \frac{1}{\sqrt[3]{a^2}},\ \frac{1}{\sqrt[3]{a}},\ 1,\ \sqrt[3]{a},\ \sqrt[3]{a^2},\ \sqrt[3]{a^3},$$

$\sqrt[3]{a^4}$, &c.

Or thus,

$$\frac{1}{a^{\frac{4}{3}}},\ \frac{1}{a},\ \frac{1}{a^{\frac{2}{3}}},\ \frac{1}{a^{\frac{1}{3}}},\ 1,\ a^{\frac{1}{3}},\ a^{\frac{2}{3}},\ a^{\frac{3}{3}},\ a^{\frac{4}{3}},\ \&c.$$

or thus, $a^{-\frac{4}{3}}$, a^{-1}, $a^{-\frac{2}{3}}$, $a^{-\frac{1}{3}}$, a^0, $a^{\frac{1}{3}}$, $a^{\frac{2}{3}}$, a^1, a^4, &c. and the ſame is equally true of any of the other ſeries.

Cor. 1. *The powers of any quantity are a ſet of geometrical proportionals from* 1; *and their indices, a ſet of arithmetic proportionals from* 0.

thus, powers $1,\ a,\ a^2,\ a^3,\ a^4,$ ∺ increaſing.
indices $0,\ 1,\ 2,\ 3,\ 4,$ ∸ increaſing.
alſo, powers $1,\ \frac{1}{a},\ \frac{1}{aa},\ \frac{1}{a^3},\ \frac{1}{a^4},$ ∺ decreaſ.
indices $0, -1, -2, -3, -4,$ ∸ decreaſ.

Cor. 2. *Hence the double, triple, quadruple, &c. the index of any quantity, is the index of the ſquare, cube, biquadrate, &c. of that quantity.*

Cor. 3. *Hence alſo, the index of the product of any two powers (whole or fracted) of any quantity, is equal to the ſum of the indices of theſe powers. And therefore to multiply any two powers together, is to add their indices. Thus* $a^2 \times a^3 = a^5$, $a^2 \times a^{-3} = a^{-1}$, &c.

Cor. 3. *The index of the quotient of two powers dividing one another, is equal to the index of the dividend — the index of the diviſor; whatever the indices be. And therefore, to divide by powers, is to ſubtract their indices. Thus* $\frac{a^3}{a} = a^2$, and $\frac{a^2}{a^5} = a^{-3}$. *Alſo* $\frac{a^3}{a^{-2}} = a^5$, &c.

Cor. 4. *Any power is taken out of the denominator, and put into the numerator, by changing the ſign of the index: and the contrary. Thus*

$$\frac{1}{a^3} = a^{-3},\quad \frac{b}{a^2} = ba^{-2}.\quad \textit{Alſo}$$

$$\frac{a^2}{b^3} = a^2\, b^{-3} = \frac{1}{a^{-2} b^3},\ \&c.$$

Cor. 5. *In fractional indices, the numerator ſhews the power, and the denominator the root.*

Schol. In all the following problems, it will be the beſt way to reduce the ſurds to fractional indices.

PROBLEM XX.

To reduce a rational quantity to the form of a ſurd.

RULE.

Multiply the index of the quantity, by the index of the ſurd root given; to which ſet the radical ſign, or index of the ſurd.

Ex. 1.

Reduce 6 *to the form of* $\sqrt{5}$.

Here $6^{1\times 2}$ or $6^2 = 36$, and $\sqrt{36}$ is that required.

Ex. 2.

Reduce a *to the form of* $\sqrt[3]{b}$.

Here $a^{1\times 3} = a^3$, and $\sqrt[3]{a^3}$ is the anſwer.

Ex. 3.

Reduce $a+b$ *to the form of* $\sqrt{bc}$.

Anſw. $\sqrt{\overline{a+b}^2}$, or $\sqrt{aa+2ab+bb}$.

Ex. 4.

Reduce $\frac{a}{b\sqrt{c}}$ *to the form of* $\sqrt{d}$.

Anſ. $\sqrt{\frac{aa}{bbc}}$ is of the form $\sqrt{d}$.

PROBLEM XXI.

To reduce quantities of different indexes, to other equal ones, that ſhall have a common index given.

RULE.

Divide the indexes of the quantities by the given index; the quotients will be the new indexes for theſe quantities. Over theſe quantities with their new indices, place the index given.

Ex. 1.

Reduce $12^{\frac{1}{4}}$ *and* $7^{\frac{1}{6}}$ *to the common index* $\frac{1}{2}$.

$$\left.\begin{array}{l}\frac{1}{2}\Big)\frac{1}{4}\Big(\frac{1}{2}\text{ firſt index.}\\ \frac{1}{2}\Big)\frac{1}{6}\Big(\frac{1}{3}\text{ ſecond index}\end{array}\right\}\text{then }\overline{12^{\frac{1}{2}}}|^{\frac{1}{2}}\text{ and }\overline{7^{\frac{1}{3}}}|^{\frac{1}{2}}\text{ are the quantities required.}$$

Ex. 2.

Reduce a^2 and $b^{\frac{3}{2}}$, to the common index $\frac{1}{3}$.

$$\left.\begin{array}{l}\frac{1}{3}\Big)\frac{2}{1}\Big(6\text{ firſt index.}\\ \frac{1}{3}\Big)\frac{3}{2}\Big(\frac{9}{2}\text{ ſec. index.}\end{array}\right\}\text{then }\overline{a^6}|^{\frac{1}{3}}\text{, and }\overline{b^{\frac{9}{2}}}|^{\frac{1}{3}}\text{ are the ſurds.}$$

PROBLEM XXII.

To reduce quantities of different indices, to others equal to them, that ſhall have the leaſt common index.

RULE.

Reduce the indices of the given quantities, to a common denominator, in the leaſt terms. Then involve each quantity to the power of its numerator; and take the root denoted by the common denominator.

Ex. 1.

Reduce $b^{\frac{1}{4}}$ and $c^{\frac{1}{6}}$ to the leaſt common index.

$\frac{1}{4}$ and $\frac{1}{6}$ are $= \frac{3}{12}$ and $\frac{2}{12}$. Therefore

$$b^{\frac{1}{4}} = b^{\frac{3}{12}}, \text{ and } c^{\frac{1}{6}} = c^{\frac{2}{12}}.$$

or $b^{\frac{1}{4}}$ and $c^{\frac{1}{6}}$ become $\overline{b^3}|^{\frac{1}{12}}$ and $\overline{c^2}|^{\frac{1}{12}}$, or

$bbb^{\frac{1}{12}}$ and $\overline{cc}|^{\frac{1}{12}}$.

Ex. 2.

Let $b^{\frac{2}{3}}$ and $\overline{dc}|^{\frac{2}{9}}$ be given.

$\frac{2}{3}$ and $\frac{2}{9}$ are reduced to $\frac{6}{9}$ and $\frac{2}{9}$.

Therefore $b^{\frac{2}{3}}$ and $\overline{dc}^{\frac{2}{9}}$ become $b^{\frac{6}{9}}$ and $\overline{dc}^{\frac{2}{9}}$, or $\overline{b^3}|^{\frac{2}{9}}$ and $\overline{dc}|^{\frac{2}{9}}$, or $\overline{b^6}|^{\frac{1}{9}}$ and $\overline{ddcc}|^{\frac{1}{9}}$.

Ex. 3.

Let $\sqrt{a+b}$, and $\sqrt[3]{aa-xx}$ be propoſed.

Theſe are $\overline{a+b}^{\frac{1}{2}}$ and $\overline{aa-xx}^{\frac{1}{3}}$. The indices are reduced to $\frac{3}{6}$ and $\frac{2}{6}$. Therefore the ſurds become $\overline{\overline{a+b}^3}|^{\frac{1}{6}}$ and $\overline{\overline{aa-xx}^2}|^{\frac{1}{6}}$; or $\overline{a^3+3aab+3abb+b^3}|^{\frac{1}{6}}$, and $\overline{a^4-2a^2xx+x^4}|^{\frac{1}{6}}$, or $\sqrt[6]{a^3+3aab+3abb+b^3}$, and $\sqrt[6]{a^4-2a^2xx+x^4}$.

PROBLEM XXIII.

To reduce ſurds to their moſt ſimple terms.

RULE.

Divide by the greateſt power contained in it, and ſet the root before the ſurd containing the remaining quantities.

Ex.

Ex. 1.

Reduce $\sqrt{48}$ *to a ſimpler form.*

$\sqrt{48}=\sqrt{3\times16}=4\sqrt{3}$ the ſurd required.

Ex. 2.

Let $\sqrt{64aabc}$ *be propoſed.*

$\sqrt{64aa}=8a$. Then $\sqrt{64aabc}=8a\sqrt{bc}$.

Ex. 3.

Reduce $\overline{a^3x-a^2x^2}|^{\frac{1}{2}}$.

Here $\sqrt{aa}=a$, and the ſurd becomes $a\times\overline{ax+xx}^{\frac{1}{2}}$, or $a\sqrt{ax-xx}$.

Ex. 4.

Let $\sqrt{\frac{a^3b-4aabb+4ab^3}{cc}}$ *be given.*

The ſurd is $\sqrt{\frac{aa-4ba+4bb}{cc}\times ab}$. And $\sqrt{\frac{aa-4ba+4bb}{cc}}=\frac{a-2b}{c}$. Therefore the ſurd becomes $\frac{a-2b}{c}\times\sqrt{ab}$, or $\frac{2-2b}{c}\sqrt{ab}$.

Ex. 5.

Given $\sqrt[3]{\frac{27a^4b^3}{8b-8a}}=\sqrt[3]{\frac{27a^3b^3\times a}{8\times\overline{b-a}}}$.

reduced, becomes $\frac{2ab}{2}\sqrt[3]{\frac{a}{b-a}}$.

PROBLEM XXIV.

To find whether two ſurds are commenſurable or not.

RULE.

Reduce them to the leaſt common index, and the quantities to a common denominator, if fractions, except when like terms are commenſurable. Then divide them by the greateſt common diviſor, (or by ſuch a one as will give one quotient rational;) then if both quotients be rational, the ſurds are commenſurable; otherwiſe not.

Ex. 1.

Let $\sqrt{18}$ *and* $\sqrt{8}$ *be propoſed.*

Theſe are $\sqrt{2 \times 9}$ and $\sqrt{2 \times 4}$. Divide by 2, and the quotients are $\sqrt{9}$, and $\sqrt{4}$; that is, 3 and 2; therefore they are commenſurable.

Ex. 2.

Let the ſurds be $\sqrt{\frac{50}{16}}$ *and* $\sqrt{\frac{72}{25}}$.

Theſe are $\frac{\sqrt{50}}{4}$ and $\frac{\sqrt{72}}{5}$. Divide by 2, and the quotients are $\sqrt{25}$ and $\sqrt{36}$, that is 5 and 6; and the ſurds become $\frac{5}{4}\sqrt{2}$ and $\frac{6}{5}\sqrt{2}$, and are therefore commenſurable, being as $\frac{5}{4}$ to $\frac{6}{5}$.

Ex. 3.

Let $\sqrt{48}$ *and* $\sqrt{8}$ *be propoſed.*

Divide by 8, the quotients are $\sqrt{6}$ and $\sqrt{1}$ or 1; therefore they are incommenſurable.

Ex. 4.

Let $\overline{\frac{b}{c}}\big|^{\frac{1}{2}}$ *and* $\overline{\frac{c}{b}}\big|^{\frac{3}{2}}$ *be given.*

Here $\overline{\frac{b}{c}}\big|^{\frac{1}{2}}$ and $\overline{\frac{c}{b}}\big|^{\frac{3}{2}}$ are reduced to $\overline{\frac{b}{c}}\big|^{\frac{1}{2}}$ and $\overline{\frac{c^3}{b^3}}\big|^{\frac{1}{2}}$, and theſe to $\overline{\frac{b^4}{cb^3}}\big|^{\frac{1}{2}}$ and $\overline{\frac{c^4}{cb^3}}\big|^{\frac{1}{2}}$. Divide by $\frac{1}{cb^3}$, and the quotients are $\overline{b^4}|^{\frac{1}{2}}$ and $\overline{c^4}|^{\frac{1}{2}}$, that is, bb and cc; therefore the ſurds are commenſurable.

Ex. 5.

Suppoſe $\sqrt{a^4+aabb}$ *and* $\sqrt{aabb+b^4}$

Theſe are $\sqrt{aa\times\overline{aa+bb}}$, and $\sqrt{bb\times\overline{aa+bb}}$. Therefore dividing by $aa+bb$, the quotients are $\sqrt{aa}$, and $\sqrt{bb}$, or a and b, and therefore they are commenſurable.

Ex. 6.

Let $\sqrt{\frac{16aa}{14b}}$ *and* $\sqrt{\frac{9aa}{8b}}$ *be given.*

That is, $\frac{4a}{\sqrt{14b}}$ and $\frac{3a}{\sqrt{8b}}$. Divide the denominators by 2, then they are reduced to $\frac{4a}{\sqrt{7b}}$ and $\frac{3a}{\sqrt{4b}}$ or $\frac{3a}{2\sqrt{b}}$, and are therefore incommenſurable.

PROBLEM XXV.

To add ſurd quantities together.

RULE.

Reduce quantities with unlike indexes, to thoſe of like indexes.

Alſo reduce fractions to a common denominator, or elſe to others that have rational denominators (or numerators).

Then reduce the quantities to the ſimpleſt terms (Prob. 23.) This being done; if the ſurd part be the ſame in all, annex it to the ſum of the rational parts, with the ſign (×) of multiplication.

If the ſurd part is not the ſame in all, the quantities can only be added by the ſigns + and —.

Ex. 1.

Add $\sqrt{6}$ *to* $2\sqrt{6}$.

The ſum is $\overline{1+2} \times \sqrt{6}$ or $3\sqrt{6}$.

Ex. 2.

Add $\sqrt{8}$ *to* $\sqrt{50}$.

$\sqrt{8} = 2\sqrt{2}$, and $\sqrt{50} = 5\sqrt{2}$, and the ſum $= \overline{2+5} \times \sqrt{2} = 7\sqrt{2} = \sqrt{98}$.

Ex. 3.

Add $\sqrt[3]{500}$ *to* $\sqrt[3]{108}$.

$\sqrt[3]{500} = \sqrt[3]{4 \times 125} = 5\sqrt[3]{4}$. And $\sqrt[3]{108} = \sqrt[3]{4 \times 27} = 3\sqrt[3]{4}$. Therefore the ſum $= \overline{5+3} \times \sqrt[3]{4} = 8\sqrt[3]{4}$.

Ex. 4.

Add $\sqrt{48a^4b}$ *to* $\sqrt{3aab^3}$.

They are reduced to $4aa\sqrt{3b}$ and $ab\sqrt{3b}$. And the ſum $= \overline{4aa+ab} \times \sqrt{3b}$.

Ex. 5.

Given $\sqrt{4a}$ *and* $\sqrt[4]{a^6}$.

$\sqrt{4a} = \overline{4a}^{\frac{1}{2}} = \overline{4a}^{\frac{2}{4}} = \overline{16aa}^{\frac{1}{4}} = \sqrt[4]{16aa} = 2\sqrt[4]{aa}$. And $\sqrt[4]{a^6} = a\sqrt[4]{aa}$. And their ſum $= \overline{a+2} \times \sqrt[4]{aa} = \overline{a+2} \times \sqrt{a}$.

Ex. 6.

Add $\sqrt{\frac{24}{25}}$ *to* $\sqrt{\frac{2}{3}}$.

Theſe reduced to a common denominator, become $\sqrt{\frac{72}{75}}$ and $\sqrt{\frac{50}{75}}$, or $\sqrt{\frac{2 \times 36}{75}}$ and $\sqrt{\frac{2 \times 25}{75}}$, that is, $6\sqrt{\frac{2}{75}}$ and $5\sqrt{\frac{2}{75}}$, whoſe ſum is $11\sqrt{\frac{2}{75}}$.

Or thus,

Here $\sqrt{\frac{24}{25}} = \frac{\sqrt{24}}{5} = \frac{\sqrt{4 \times 6}}{5} = \frac{2}{5}\sqrt{6}$; Alſo $\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$. And their ſum $= \overline{\frac{2}{5} + \frac{1}{3}} \times \sqrt{6} = \frac{11}{15}\sqrt{6} = \frac{11}{5}\sqrt{\frac{2}{3}}$

Ex. 7.

Add $\sqrt[3]{\frac{1}{4}}$ *to* $\sqrt[3]{\frac{16}{27}}$.

Theſe become $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[3]{\frac{64}{108}}$, or $\sqrt[3]{\frac{1}{4}}$

and $4\sqrt[3]{\frac{1}{4\times27}}$ that is $\sqrt[3]{\frac{1}{4}}$ and $\frac{4}{3}\sqrt[3]{\frac{1}{4}}$; whoſe ſum is $\overline{1+\frac{4}{3}}\times\sqrt[3]{\frac{1}{4}}=\frac{7}{3}\sqrt[3]{\frac{1}{4}}$.

Or thus,

$$\sqrt[3]{\frac{1}{4}}=\sqrt[3]{\frac{27}{108}}=3\sqrt[3]{\frac{1}{4\times27}}=\frac{3}{3}\sqrt[3]{\frac{1}{4}};$$

and $\sqrt[3]{\frac{16}{27}}=\sqrt[3]{\frac{64}{108}}=4\sqrt[3]{\frac{1}{4\times27}}=\frac{4}{3}\sqrt[3]{\frac{1}{4}}$:

And the ſum $=\frac{7}{3}\sqrt[3]{\frac{1}{4}}$.

Ex. 8.

Add $\overline{\frac{b}{c}}\Big|^{\frac{1}{2}}$ *to* $\overline{\frac{c}{b}}\Big|^{\frac{3}{2}}$.

Theſe are reduced to $\overline{\frac{b^4}{cb^3}}\Big|^{\frac{1}{2}}$ and $\overline{\frac{c^4}{cb^3}}\Big|^{\frac{1}{2}}$ or $\frac{bb}{b}\sqrt{\frac{1}{bc}}$ and $\frac{cc}{b}\sqrt{\frac{1}{bc}}$: And their ſum is

$$\frac{bb+cc}{b}\sqrt{\frac{1}{bc}}=\frac{bb+cc}{b\sqrt{bc}}.$$

Ex. 9.

Add $\sqrt{ccddaa-ccddxx\sqrt{2}}$ *to* $\sqrt{d^4aa-d^4xx\sqrt{2}}$.

They are reduced to $cd\sqrt{aa-xx\sqrt{2}}$, and $dd\sqrt{aa-xx\sqrt{2}}$, and the ſum is $\overline{cd+dd}\times\sqrt{aa-xx\sqrt{2}}$.

Ex. 10.

$$\begin{array}{ll}
\text{To} & 2\sqrt[3]{aa}-\sqrt{a^3}+\sqrt{13}. \\
\text{Add} & \sqrt[3]{aa}+2\sqrt{a}-\sqrt{7}. \\
\hline
\text{Sum} & 3\sqrt[3]{aa}-\sqrt{a^3}+\sqrt{13}+2\sqrt{a}-\sqrt{7}.
\end{array}$$

PRO-

PROBLEM XXVI.

To ſubtract ſurd quantities.

RULE.

Reduce, as in the laſt rule; then ſubtract the rational quantities, and annex the difference to the common ſurd, with the ſign (×) of multiplication.

EXAMPLES.

1. Subtract $\sqrt{6}$ from $2\sqrt{6}$, the remainder is $\overline{2-1} \times \sqrt{6} = \sqrt{6}$.

2. $\sqrt{50} - \sqrt{8} = 5\sqrt{2} - 2\sqrt{2} = 3\sqrt{2}$.

3. $\sqrt[3]{500} - \sqrt[3]{108} = 5\sqrt[3]{4} - 3\sqrt[3]{4} = 2\sqrt[3]{4}$.

4. $\sqrt{48a^4b} - \sqrt{3aab^3} = 4aa\sqrt{3b} - ab\sqrt{3b}$
$= \overline{4aa - ab} \times \sqrt{3b}$.

5. $\sqrt[4]{a^6} - \sqrt{4a} = \sqrt{a^3} - \sqrt{4a} = a\sqrt{a} -$
$2\sqrt{a} = \overline{a - 2} \times \sqrt{a}$.

6. $\sqrt{\frac{24}{25}} - \sqrt{\frac{2}{3}} = \sqrt{\frac{72}{75}} - \sqrt{\frac{50}{75}} = 6\sqrt{\frac{2}{75}}$
$- 5\sqrt{\frac{2}{75}} = \sqrt{\frac{2}{75}}$.

or $\sqrt{\frac{24}{25}} - \sqrt{\frac{2}{3}} = \sqrt{\frac{24}{25}} - \sqrt{\frac{6}{9}} =$
$\frac{2}{5}\sqrt{6} - \frac{1}{3}\sqrt{6} = \frac{1}{15}\sqrt{6}$.

7. $\sqrt[3]{\frac{16}{27}} - \sqrt[3]{\frac{1}{4}} = \frac{4}{3}\sqrt[3]{\frac{1}{4}} - \sqrt[3]{\frac{1}{4}} =$
$\frac{1}{3}\sqrt[3]{\frac{1}{4}}$. Or thus, $\sqrt[3]{\frac{16}{27}} - \sqrt[3]{\frac{1}{4}} = \sqrt[3]{\frac{64}{4 \times 27}}$
$- \sqrt[3]{\frac{1}{4}} = \frac{4}{3}\sqrt[3]{\frac{1}{4}} - \sqrt[3]{\frac{1}{4}} = \frac{1}{3}\sqrt[3]{\frac{1}{4}}$.

8. $\left.\overline{\frac{b}{c}}\right|^{\frac{1}{2}} - \left.\overline{\frac{c}{b}}\right|^{\frac{3}{2}} = \frac{b}{1} \times \left.\overline{\frac{1}{bc}}\right|^{\frac{1}{2}} - \frac{cc}{b} \times \left.\overline{\frac{1}{bc}}\right|^{\frac{1}{2}} = \frac{bb-cc}{b} \times \left.\overline{\frac{1}{bc}}\right|^{\frac{1}{2}}.$

9. $\sqrt{ccddaa-ccddxx} - \sqrt{d^4aa-d^4xx} = cd\sqrt{aa-xx} - dd\sqrt{aa-xx} = \overline{cd-dd} \times \sqrt{aa-xx}.$

10. *From* $2\sqrt[3]{aa} - \sqrt{a^3} + \sqrt{13}.$

take $\sqrt[3]{aa} + 2\sqrt{a} - \sqrt{7}.$

rem. $\sqrt[3]{aa} - \sqrt{a^3} + \sqrt{13} - 2\sqrt{a} + \sqrt{7}.$

PROBLEM XXVII.

To multiply ſurds.

RULE.

Surds by ſurds; if they have not the ſame index already, reduce them to the ſame; then multiply the quantities under the common index.

Ex. 1.

Multiply $\sqrt{5}$ *by* $\sqrt{3}$.

$\sqrt{5}$

$\sqrt{3}$

the product $\sqrt{15}$.

Ex. 2.

Multiply $\sqrt{\frac{2ab}{3c}}$ *by* $\sqrt{\frac{9ad}{2b}}$.

product $= \sqrt{\frac{18aabd}{6bc}} = \sqrt{\frac{3aad}{c}}$.

Ex. 3.

Multiply $\sqrt{d}$ *by* $\sqrt[3]{ab}$.

Reduced to $\overline{d^6}|^{\frac{1}{6}}$ and $\overline{aabb}|^{\frac{1}{6}}$; the product $= \overline{a^2b^2d^6}|^{\frac{1}{6}} = \sqrt[3]{aabbd^3}$.

Ex. 4.

Multiply $a^{\frac{2}{3}}$ *by* $a^{\frac{1}{4}}$.

Product $a^{\frac{2}{3}} \times a^{\frac{1}{4}} = a^{\frac{2}{3}+\frac{1}{4}} = a^{\frac{11}{12}}$.

2 RULE.

A ſurd by a rational quantity; connect them with the ſign (×) of multiplication; or elſe reduce the rational quantity to the form of that ſurd, and multiply by Rule 1.

Ex. 5.

Multiply $\sqrt{4a - 3x}$ *by* $2a$.

The product is $2a \times \sqrt{4a - 3x}$

Or $2a = \sqrt{4aa}$, then the product $= \sqrt{16a^3 - 12aax}$.

3 RULE.

When rational quantities are annexed to ſurds; multiply the rational by the rational, and the ſurd by the ſurd.

Ex. 6.

Multiply $\frac{a}{2b}\sqrt{a-x}$ *by* $\overline{c-d}\,\sqrt{ax}$.

The product $= \frac{a \times \overline{c-d}}{2b}\sqrt{\overline{a-x} \times ax} =$

$$\frac{ac-ad}{2b}\sqrt{aax-axx}$$

Ex. 7.

Multiply $\frac{a}{b}\sqrt{ax}$ *by* $\overline{b-x}\sqrt[3]{\frac{ax^3}{b}}$.

Here $\frac{a}{b}\sqrt{ax} = \frac{a}{b}\times\overline{ax}|^{\frac{3}{6}} = \frac{a}{b}\times\overline{a^3x^3}|^{\frac{1}{6}}$.

And $\overline{b-x}\sqrt{\frac{ax^3}{b}} = \overline{b-x}\times\overline{\frac{ax^3}{b}}|^{\frac{2}{6}} = \overline{b-x}\times\overline{\frac{aax^6}{bb}}|^{\frac{1}{6}}$. Therefore

$\frac{a}{b}\times\overline{a^3x^3}|^{\frac{1}{6}}\times$ into $\overline{b-x}\times\overline{\frac{aax^6}{bb}}|^{\frac{1}{6}} = \frac{ab-ax}{b}\times\overline{\frac{a^5x^9}{bb}}|^{\frac{1}{6}} = \frac{ab-ax}{b}\sqrt[6]{\frac{a^5x^9}{bb}} = \frac{ab-ax}{b}x\sqrt[6]{\frac{a^5x^3}{bb}}$ the product.

Ex. 8.

$$\begin{array}{rl}
\textit{Multiply} & a+\sqrt{b}-d \\
\textit{by} & a-\sqrt{b} \\
\hline
 & aa+a\sqrt{b}-ad \\
 & \quad -a\sqrt{b}-b+d\sqrt{b} \\
\hline
\text{product} & aa-ad-b+d\sqrt{b} \\
\hline
\end{array}$$

Ex. 9.

$$\begin{array}{rl}
\textit{Multiply} & 2a-3a\sqrt{d} \\
\textit{by} & 3c-2c\sqrt{d} \\
\hline
 & 6ac-9ac\sqrt{d} \\
 & \quad -4ac\sqrt{d}+bac\sqrt{dd} \\
\hline
\text{product} & 6ac-13ac\sqrt{d}+6acd
\end{array}$$

Ex. 10.

$$\text{Multiply } \sqrt{a-\sqrt{b-\sqrt{3}}}$$

$$\text{by } \sqrt{a+\sqrt{b-\sqrt{3}}}$$

$$\text{product } \sqrt{aa-a\sqrt{b-\sqrt{3}}+a\sqrt{b-\sqrt{3}}-b+\sqrt{3}},$$

$$\text{or } \sqrt{aa-b+\sqrt{3}}.$$

Schol. If impoſſible or imaginary roots be multiplied together, they always produce —, otherwiſe a real product would be raiſed from impoſſible factors, which is abſurd. Thus, $\sqrt{-a}\times\sqrt{-b}=\sqrt{-ab}$, and $\sqrt{-a}\times-\sqrt{-b}=-\sqrt{-ab}$, &c. Alſo $\sqrt{-a}\times\sqrt{-a}=-a$, and $\sqrt{-a}\times-\sqrt{-a}=+a$, &c.

PROBLEM XXVIII.

To divide ſurds.

1 RULE.

In ſurds of the ſame ſimple quantity; ſubtract their indices from each other.

Ex. 1.

$$\text{Divide } a^{\frac{2}{3}} \text{ by } a^{\frac{1}{4}}.$$

$$\text{quotient } a^{\frac{2}{3}-\frac{1}{4}} = a^{\frac{5}{12}}.$$

Ex. 2.

$$\text{Divide } a^{\frac{1}{n}} \text{ by } a^{\frac{1}{m}}.$$

$$\text{quotient } a^{\frac{1}{n}-\frac{1}{m}} = a^{\frac{m-n}{mn}}.$$

2 RULE.

If they be different quantities, reduce them to the ſame index, if they are not ſo already. Then divide the quantities under the common index.

Ex. 3.

Divide $\sqrt{15}$ *by* $\sqrt{5}$.

$5)\ 15\ (\sqrt{3}$ the quotient.

Ex. 4.

Divide $\sqrt{\frac{3aad}{c}}$, *by* $\sqrt{\frac{2ab}{3c}}$.

$$\sqrt{\frac{2ab}{3c}}\Big)\ \sqrt{\frac{3aad}{c}}\ \Big(\sqrt{\frac{9ad}{2b}}$$, quotient.

Ex. 5.

Divide $\sqrt[6]{aabbd^3}$ *by* $\sqrt{d}$.

$\sqrt{d} = \sqrt[6]{d^3}$. $d^3)\ aabbd^3\ (\sqrt[6]{aabb} = \sqrt[3]{ab}$, quot.

3 RULE.

If rational quantities are annexed; divide rational quantities by rational quantities, and ſurds by ſurds.

Ex. 6.

Divide $\sqrt{16a^3 - 12aax}$ *by* $2a$.

quotient $\frac{1}{2a}\sqrt{16a^3 - 12aax} = \sqrt{\frac{16a^3 - 12aax}{4aa}} = \sqrt{4a - 3x}$.

Ex. 7.

Divide $\frac{ac-ad}{2b}\sqrt{aax-axx}$ by $\frac{a}{2b}\sqrt{a-x}$.

$\frac{a}{2b}\Big)\frac{ac-ad}{2b}\Big(\frac{c-d}{1}$. $a-x)\ aax-axx\ (\sqrt{ax}$.

Then $\overline{c-d}\times\sqrt{ax}$ = quotient.

Ex. 8.

Divide $\frac{ab-ax}{b}x\sqrt[6]{\frac{a^5x^3}{bb}}$ by $\overline{b-x}\sqrt[3]{\frac{ax^3}{b}}$.

$$\sqrt[3]{\frac{ax^3}{b}}=\sqrt[6]{\frac{aax^6}{b^2}}.$$

$b-x\Big)\frac{ab-ax}{b}x\Big(\frac{ax}{b}$. $\frac{aax^6}{b^2}\Big)\frac{a^5x^3}{bb}\Big(\frac{a^3}{x^3}$.

Then the quotient

$$=\frac{ax}{b}\sqrt[6]{\frac{a^3}{x^3}}=\frac{ax}{b}\sqrt{\frac{a}{x}}=\frac{a}{b}\sqrt{ax}.$$

Ex. 9.

$$\begin{array}{llll}
a-\sqrt{b})\ & aa-ad-b+d\sqrt{b}\ (a+\sqrt{b}-d & & \\
 & aa-a\sqrt{b} & & \text{quotient.} \\
\cline{2-2}
 & +a\sqrt{b} & & \\
 & +a\sqrt{b}-b & & \\
\cline{2-2}
 & 0 \qquad 0 & -ad & \\
 & & -ad+d\sqrt{b} & \\
\cline{3-3}
 & & 0 &
\end{array}$$

Ex. 10.

$$\begin{array}{l} aa+a\sqrt{bc})\ a^3b-abbc\ (ab-b\sqrt{bc} \\ \qquad\qquad\ \ a^3b+a^2b\sqrt{bc} \\ \hline \qquad\qquad\qquad -a^2b\sqrt{bc}-abbc \\ \qquad\qquad\qquad -a^2b\sqrt{bc}-abbc \\ \hline \qquad\qquad\qquad\qquad 0 \end{array}$$

Ex. 11.

Divide $\sqrt{aa-b+\sqrt{3}}$ by $\sqrt{a-\sqrt{b-\sqrt{3}}}$.

$$\begin{array}{l} a-\sqrt{b-\sqrt{3}})\ aa-b+\sqrt{3}\ (\sqrt{a+\sqrt{b-\sqrt{3}}} \\ \qquad\qquad\quad aa-a\sqrt{b-\sqrt{3}} \\ \hline \qquad\qquad\qquad +a\sqrt{b-\sqrt{3}}-b+\sqrt{3} \\ \qquad\qquad\qquad +a\sqrt{b-\sqrt{3}}-b+\sqrt{3} \\ \hline \qquad\qquad\qquad\qquad\quad 0 \end{array}$$

Ex. 12.

Divide $\sqrt{bbca}+\sqrt{aab}-bc-\sqrt{abc}$
by $\sqrt{bc}+\sqrt{a}$

$$\begin{array}{l} \sqrt{bc}+\sqrt{a})\sqrt{bbca}+\sqrt{aab}-bc-\sqrt{abc}(\sqrt{ba}-\sqrt{bc} \\ \qquad\qquad\ \ \sqrt{bbca}+\sqrt{aab} \\ \hline \qquad\qquad\qquad\qquad 0\quad -bc-\sqrt{abc} \\ \qquad\qquad\qquad\qquad\quad\ \ -bc-\sqrt{abc} \\ \hline \qquad\qquad\qquad\qquad\qquad\quad 0 \end{array}$$

4 RULE.

When the quantities will not divide, ſet them down in form of a fraction.

Ex. 13.

Divide $bcd + \sqrt{abb}$ *by* $\sqrt{ab} - \sqrt{4bc}$

The quotient is $\dfrac{\sqrt{bcd} + \sqrt{abb}}{\sqrt{ab} - \sqrt{4bc}}$.

PROBLEM XXIX.

To involve ſurd quantities to any power.

1 RULE.

Multiply the index of the quantity, by the index of the power to be raiſed.

Ex. 1.

Let $\sqrt{2}$ *be cubed.*

$\sqrt{2} = \overline{2}|^{\frac{1}{2}}$. Then $2^{\frac{1}{2}\times 3}$ or $2^{\frac{3}{2}}$ is the cube, that is $\sqrt{2^3}$ or $\sqrt{8}$ = the cube of $\sqrt{2}$.

Ex. 2.

What is the ſquare of $3\sqrt[3]{bcc}$.

$3\sqrt[3]{bcc} = 3 \times \overline{bcc}|^{\frac{1}{3}}$. Its ſquare $=$ $9 \times \overline{bcc}|^{\frac{2}{3}} = 9\sqrt[3]{bbc^4} = 9c\sqrt[3]{bbc}$.

Ex. 3.

What is the cube of $a\sqrt{a-x}$.

$a\sqrt{a-x} = a^1 \times \overline{a-x}^{1}$; cubed it is $a^3 \times \overline{a-x}^{\frac{3}{2}}$ that is, the cube $= a\sqrt{a^3 - 3a^2x + 3ax^2 - x^3}$

Ex. 4.

What is the 4th power of $\frac{a}{2b}\sqrt{\frac{2a}{c-b}}$.

Here $\frac{a}{2b}\sqrt{\frac{2a}{c-b}} = \frac{a}{2b} \times \overline{\frac{2a}{c-b}}\Big|^{\frac{1}{2}}$. And its 4th power is $\overline{\frac{a}{2b}}\Big|^{4} \times \overline{\frac{2a}{c-b}}\Big|^{\frac{4}{2}} = \frac{a^4}{16b^4} \times \overline{\frac{2a}{c-b}}\Big|^{2} = \frac{4a^6}{16b^4 \times \overline{c-b}|^{2}} = \frac{a^6}{4b^4 \times \overline{cc-2bc+bb}}$.

2 RULE.

If quantities are to be involved to a power denoted by the index of the ſurd root; take away the radical ſign.

Ex. 5.

Let $\sqrt{\frac{4ab}{cc}}$ *be ſquared.*

Its ſquare is $\frac{4ab}{cc}$.

Ex. 6.

What is the cube of $\sqrt[3]{a^3-b^3+3b\sqrt{abb}}$.

Anſwer, $a^3-b^3+3b\sqrt{abb}$.

3 RULE.

Compound ſurds are involved as integers, obſerving the rule of multiplication of ſurds.

Ex. 7.

Let $3+\sqrt{5}$ *be ſquared.*

$$\begin{array}{r} 3+\sqrt{5} \\ 3+\sqrt{5} \\ \hline 9+3\sqrt{5} \\ +3\sqrt{5}+5 \\ \hline \text{the ſquare}\quad 14+6\sqrt{5} \end{array}$$

Ex. 8.

Let $a-\sqrt{b}$ *be cubed.*

$$\begin{array}{rl}
 & a-\sqrt{b} \\
 & a-\sqrt{b} \\
\hline
 & aa-a\sqrt{b} \\
 & \quad -a\sqrt{b}+b \\
\hline
 & aa-2a\sqrt{b}+b \\
 & a-\sqrt{b} \\
\hline
 & a^3-2aa\sqrt{b}+ab \\
 & \quad -aa\sqrt{b}+2ab-b\sqrt{b} \\
\hline
\text{the cube} & a^3-3aa\sqrt{b}+3ab-b\sqrt{b}. \\
\hline
\end{array}$$

PROBLEM XXX.

To extract any root of a ſurd.

RULE.

Divide the index of the quantity or quantities, by the index of the root to be extracted.

Ex. 1.

Extract the ſquare root of a^3.

The root $= a^{\frac{3}{2}} = \sqrt{a^3}$.

Ex. 2.

Extract the cube root of ab^2.

The root is $a^{\frac{1}{3}}b^{\frac{2}{3}} = \sqrt[3]{abb}$.

Ex. 3.

What is the 4th root of $3aa$.

The root is $a^{\frac{2}{4}}\sqrt[4]{3} = a^{\frac{1}{2}}\sqrt[4]{3} = \sqrt{a} \times \sqrt[4]{3}$.

Ex. 4.

What is the cube root of $\sqrt{aa - xx}$.

The root is $\overline{aa-xx}^{\frac{1}{2}\times\frac{1}{3}} = \overline{aa-xx}^{\frac{1}{6}} = \sqrt[6]{aa-xx}.$

2 RULE.

When the index of the root to be extracted, is the ſame as the index of the power of that quantity; take away that index, and the quantity itſelf is the root.

Ex. 5.

What is the ſquare root of 3^2a^2.

Anſw. $3a$, the root.

Ex. 6.

What is the cube root of $\overline{5ax - 3xx}^3$

Anſw. $5ax - 3xx$, the root.

3 RULE.

Compound ſurds are extracted as integers, due regard being had to the operations of ſimple ſurds. When no ſuch root can be found, prefix the radical ſign.

Ex. 7.

For the ſquare root of $aa - 4a\sqrt{b} + 4b$.

$$\begin{array}{rl}
 & aa - 4a\sqrt{b} + 4b\ (a - 2\sqrt{b} \\
 & aa \\
\cline{2-2}
2a - 2\sqrt{b}) & 0 - 4a\sqrt{b} + 4b \\
 & -4a\sqrt{b} + 4b \\
\cline{2-2}
 & 0
\end{array}$$

Ex. 8.

What is the cube root of $aa - \sqrt{ax - xx}$.

Anſw. $\sqrt[3]{ax - \sqrt{ax - xx}}$, the root.

PROBLEM XXXI.

To change a binomial ſurd quantity into another.

RULE.

This reduction is performed by an equal involution, and evolution. Involve the binomial to the power denoted by the ſurd or ſurds, then ſet the radical ſign of the ſame root before it.

Ex. 1.

To transform $2 + \sqrt{3}$ *to another.*

Its ſquare, $4 + 3 + 4\sqrt{3} = 7 + 4\sqrt{3}$

the ſquare root, $\sqrt{7 + 4\sqrt{3}}$.

Ex. 2.

Reduce $\sqrt{2} + \sqrt{3}$ *to a univerſal ſurd.*

Its ſquare $2 + 3 + 2\sqrt{6} = 5 + 2\sqrt{6}$

the root $\sqrt{5 + 2\sqrt{6}}$.

Ex. 3.

Let $\sqrt{a} - 2\sqrt{x}$ *be given to reduce.*

The ſquare $a + 4x - 4\sqrt{ax}$.

the root $\sqrt{a + 4x - 4\sqrt{ax}}$.

Ex. 4.

Let $\sqrt[3]{a} + \sqrt[3]{b}$ *be given.*

The cube $a + 3\sqrt[3]{aab} + 3\sqrt[3]{abb} + b$

the root $\sqrt[3]{a + 3\sqrt[3]{aab} + 3\sqrt[3]{abb} + b}$.

Cor. $\sqrt{a}+\sqrt{b}=\sqrt{\overline{\sqrt{a}+\sqrt{b}}|^{2}}$; *and in general* $\overline{a^{\frac{1}{n}}+b^{\frac{1}{n}}} = \sqrt[n]{\overline{a^{\frac{1}{n}}+b^{\frac{1}{n}}}|^{n}}$.

PROBLEM XXXII.

To extract the ſquare root of a binomial (or reſidual) ſurd, A + B, *or* A — B; *or a trinomial,* &c.

1 RULE, *for binomials.*

Take $\sqrt{AA-BB}=D$. Then $\sqrt{A+B}=\sqrt{\frac{A+D}{2}}+\sqrt{\frac{A-D}{2}}$.

and $\sqrt{A-B}=\sqrt{\frac{A+D}{2}}-\sqrt{\frac{A-D}{2}}$.

For if $\sqrt{\frac{A+D}{2}}+\sqrt{\frac{A-D}{2}}$ be involved by Prob. 29. it will produce $A+\sqrt{AA-DD}$, that is A + B, as it ought. And $\sqrt{\frac{A+D}{2}}-\sqrt{\frac{A-D}{2}}$ will alſo produce A — B.

Ex. 1.

To extract the root of $7+\sqrt{20}$.

Here $A=7$, $B=\sqrt{20}$, and $\sqrt{AA-BB}=\sqrt{29}=D$.

Then the ſquare root of $7+\sqrt{20}=$

$\sqrt{\frac{\sqrt{7}+\sqrt{29}}{2}}+\sqrt{\frac{\sqrt{7}-\sqrt{29}}{2}}$.

Ex.

Ex. 2.

What is the ſquare root of $3-2\sqrt{2}$.

Here $\sqrt{AA-BB}=\sqrt{1}=1=D$, and $\frac{A+D}{2}=2$, $\frac{A-D}{2}=1$. And $\sqrt{3-2\sqrt{2}}=\sqrt{2}-\sqrt{1}=\sqrt{2}-1$, the root.

Ex. 3.

To extract the root of $27+\sqrt{704}$.

$\sqrt{AA-BB}=\sqrt{25}=D=5$. And the root $=\sqrt{\frac{32}{2}}+\sqrt{\frac{22}{2}}$ that is, $\sqrt{27+\sqrt{704}}=\sqrt{16}+\sqrt{11}=4+\sqrt{11}$.

Ex. 4.

What is the ſquare root of $6-2\sqrt{5}$.

Here $\sqrt{AA-BB}=\sqrt{36-20}=D=4$. And $\sqrt{\frac{A+D}{2}}=\sqrt{5}$, and $\sqrt{\frac{A-D}{2}}=1$, And the root $=\sqrt{5}-1$.

Ex. 5.

Extract the root of $\sqrt{21}+\sqrt{5}$.

$\sqrt{AA-BB}=\sqrt{16}=D=4$. And $\frac{A+D}{2}=\frac{\sqrt{21}+4}{2}$, $\frac{A-D}{2}=\frac{\sqrt{21}-4}{2}$. And the root $\sqrt{\frac{\sqrt{21}+4}{2}}+\sqrt{\frac{\sqrt{21}-4}{2}}$;

Ex. 6.

Extract the root of $aa+2x\sqrt{aa-xx}$.

Here $A=aa$, $B=2x\sqrt{aa-xx}$. Then $\sqrt{AA-BB}=\sqrt{aa-4a^2x^2+4x^4}=aa-2xx=D$. Then $\frac{A+D}{2}=aa-xx$, and $\frac{A-D}{2}=xx$, and the root $=x+\sqrt{ax-xx}$.

Ex. 7.

What is the root of $6+\sqrt{8}-\sqrt{12}-\sqrt{24}$.

Let $A=6+\sqrt{8}$, $B=\sqrt{12}+\sqrt{24}$. Then $\sqrt{AA-BB}=D=\sqrt{44+12\sqrt{8}-36-2\sqrt{12\times24}}$ $=\sqrt{8}$. $\frac{A+D}{2}=3+\sqrt{8}$, $\frac{A-D}{2}=3$. And the root $=\sqrt{3+\sqrt{8}}-\sqrt{3}$. But $\sqrt{3+\sqrt{8}}=1+\sqrt{2}$, (ſee Ex. 2.); therefore the root $=1+\sqrt{2}-\sqrt{3}$.

2 RULE, *for trinomials*, &c.

For trinomial, quadrinomial ſurds, *&c.* divide half the product of any two radicals by a third, gives the ſquare of one radical part of the root. This repeated with different quantities, will give the ſquares of all the parts of the root, to be connected by + and —. But if any quantity occur oftener than once; it muſt be taken but once.

For if $x+y+z$ be any trinomial ſurd, its ſquare will be $x^2+y^2+z^2+2xy+2xz+2yx$; then if half the product of any two rectangles as $2xy\times2xz$ (or $2x^2y^2$) be divided by ſome third $2yz$, the quotient $\frac{2x^2yz}{2yz}=xx$, muſt needs be the ſquare of one of the parts; and the like for the reſt.

Ex.

Ex. 8.

To extract the ſquare root of

$$6+\sqrt{8}-\sqrt{12}-\sqrt{24}.$$

Here $\frac{\sqrt{8}\times\sqrt{12}}{2\sqrt{24}}=1$, and $\frac{\sqrt{8}\times\sqrt{24}}{2\sqrt{12}}=\sqrt{4}=2$, and $\frac{\sqrt{12}\times\sqrt{24}}{2\sqrt{8}}=\sqrt{9}=3$. And the root is $1+\sqrt{2}-\sqrt{3}$.

Ex. 9.

To find the ſquare root of

$$12+\sqrt{32}-\sqrt{48}+\sqrt{80}-\sqrt{24}+\sqrt{40}-\sqrt{60}.$$

Here $\frac{\sqrt{32\times48}}{2\sqrt{80}}=\sqrt{\frac{24}{5}}$, this produces nothing. Again, $\frac{\sqrt{32\times48}}{2\sqrt{24}}=\sqrt{16}=4$. And $\frac{\sqrt{40\times60}}{2\sqrt{24}}=\sqrt{25}=5$; and $\frac{\sqrt{24\times40}}{2\sqrt{60}}=\sqrt{4}=2$; and $\frac{\sqrt{48\times24}}{2\sqrt{32}}=\sqrt{9}=3$; and $\frac{\sqrt{32\times80}}{2\sqrt{40}}=\sqrt{16}=4$, &c. therefore the parts of the root are $\sqrt{4}$, $\sqrt{5}$, $\sqrt{3}$, $\sqrt{2}$, $\sqrt{4}$, &c. and the root $2+\sqrt{2}-\sqrt{3}+\sqrt{5}$; for being ſquared it produces the ſurd quantity given.

Cor. 1. *In binomials, if* D *be a rational quantity, the root will conſiſt of two ſurds; and the parts of each under the radical ſign will conſiſt of a rational quantity* (D), *and a ſurd* (A).

Cor. 2. *If both* A *and* D *be rational, the root will conſiſt either of the two ſurds, or elſe of a rational part and a ſurd; which is the only caſe that is uſeful in this extraction.*

PRO-

PROBLEM XXXIII.

To extract any root (c) *of a binomial ſurd* $A+B$, *or* $A-B$.

RULE.

Let $AA-BB=D$, take Q ſuch, that $QD=n^c$, the leaſt integer power. Let $\sqrt[c]{\overline{A+B}\times\sqrt{Q}}=r$, the neareſt integer number.
Reduce $A\sqrt{Q}$ to the ſimpleſt form $p\sqrt{s}$.

Let $\dfrac{r+\frac{n}{r}}{2\sqrt{s}}=t$, the neareſt integer.

Then the root $=\dfrac{t\sqrt{s}\pm\sqrt{tts-n}}{\sqrt[2c]{Q}}$, if it can be extracted.

Note, $+$ is for the binomial $A+B$, and $-$ for the reſidual $A-B$.

Ex. 1.

What is the cube root of $\sqrt{968}+25$.

Here $D=343=7\times7\times7$. $Q\times7^3=n^3$, and $Q=1$, $n=7$. Then $\sqrt[3]{\overline{A+B}\times\sqrt{Q}}=\sqrt[3]{56+}$ $=r=4$. $A\sqrt{Q}=\sqrt{968}=22\sqrt{2}=p\sqrt{s}$, and $\sqrt{s}=\sqrt{2}$. $\dfrac{r+\frac{n}{r}}{2\sqrt{s}}=\dfrac{4+\frac{7}{4}}{2\sqrt{2}}=t=2$. And $t\sqrt{s}=2\sqrt{2}$, $\sqrt{tts-n}=\sqrt{8-7}=1$. $\sqrt[6]{Q}=1$: And the root $\dfrac{2\sqrt{2}+1}{1}=2\sqrt{2}+1$, which ſucceeds.

Ex. 2.

Extract the cube root of $68 - \sqrt{4374}$.

Here $D = 250 = 5 \times 5 \times 5 \times 2$. And $5^3 \times 2^3 = 4D = QD = n^3$, and $Q = 4$, $n = 2 \times 5 = 10$. And $\sqrt[3]{\overline{A+B} \times \sqrt{Q}} = \sqrt[3]{134 \times 2} = 6 = r$.

$A\sqrt{Q} = 136\sqrt{1} = p\sqrt{s}$, and $\sqrt{s} = 1$.

$$\frac{r + \frac{n}{r}}{2\sqrt{s}} = \frac{7\frac{2}{3}}{2} = \frac{23}{6} = 4 = t. \quad t\sqrt{s} = 4.$$

And the root

$$\frac{t\sqrt{s} - \sqrt{tts - n}}{\sqrt[6]{2}} = \frac{- \sqrt{16 - 10}}{\sqrt[6]{4}} = \frac{4 - \sqrt{6}}{\sqrt[3]{2}};$$

for its cube is $68 - 27\sqrt{6}$.

Ex. 3.

Extract the 5th root of $29\sqrt{6} + 41\sqrt{3}$.

Here $D = 3$, $n = 3$, $Q = 81$, $r = 5$, $\sqrt{s} = \sqrt{6}$, $t = 1$, $t\sqrt{s} = \sqrt{6}$, $\sqrt{tts - n} = \sqrt{3}$, $\sqrt[20]{Q} = \sqrt[10]{81} = \sqrt[5]{9}$. And the root to be tried $\frac{\sqrt{6} + \sqrt{3}}{\sqrt[5]{9}}$.

SCHOLIUM.

If the quantity be a fraction or has a common diviſor, extract the root of the denominator or of that common diviſor, ſeparately. They that would ſee the demonſtration of this rule, may conſult *Graveſande*'s or *Mac Laurin*'s Algebra. For as it ſeldom happens that ſuch quantities have a proper root; it is not worth while ſpending any more time about them.

PROBLEM XXXIV.

A compound ſurd being given, conſiſting of two, three, or more terms; which are ſurd ſquare roots: to find ſuch a multiplier or multipliers, by which multiplying the given ſurd; the product will be rational.

RULE.

Change the ſign of one of the terms in a binomial, or trinomial, or the ſigns of two terms in a quadrinomial; and by this multiply the given ſurd.

Ex. 1.

Let $a+\sqrt{3}$ *be given.*

Multiply by $a-\sqrt{3}$

product $aa-3.$

Ex. 2.

Given $\sqrt{5}-\sqrt{x}.$

Multiply by $\sqrt{5}+\sqrt{x}$

product $5-x$ rational.

Ex. 3.

Let $\sqrt{5}+\sqrt{3}-\sqrt{2}$ *be given.*

Multiply by $\sqrt{5}+\sqrt{3}+\sqrt{2}$

$$\begin{array}{l} 5+\sqrt{12}-\sqrt{10} \\ \quad +\sqrt{15}+3-\sqrt{6} \\ \qquad\qquad +\sqrt{10}+\sqrt{6}-2 \end{array}$$

product $6+2\sqrt{15}$

multiply by $-6+2\sqrt{15}$

product $60-36=24.$

Ex. 4.

$$\begin{array}{rl}
\text{There is given} & \sqrt{a}+\sqrt{b}-\sqrt{c}+\sqrt{d} \\
\text{Multiply by} & \underline{\sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{d}} \\
\text{product} & a+b-c-d+2\sqrt{ab}+2\sqrt{dc} \\
\\
\text{or} & f+2\sqrt{ab}+2\sqrt{dc}. \\
\text{multiply by} & \underline{f+2\sqrt{ab}-2\sqrt{dc}} \\
\text{product} & ff+4f\sqrt{ab}+4ab-4dc \\
\\
\text{or} & g+4f\sqrt{ab} \\
\text{multiply by} & \underline{g-4f\sqrt{ab}} \\
\text{product} & \underline{gg-16ffab}
\end{array}$$

In this proceſs f is put for the rational part $a+b-c-d$; and g for $ff+4ab-4dc$.

Cor. *A binomial becomes rational after one operation, a trinomial after two, and a quadrinomial after three,* &c.

PROBLEM XXXV.

A binomial being given, conſiſting of one or two ſurds, whoſe index or root is any power of 2; *to find a multiplier or multipliers that ſhall make it rational.*

RULE.

Multiply it by its correſponding reſidual (that is when one ſign is changed); and repeat the ſame operation, as long as there are ſurds.

Ex. 1.

Let $\sqrt{a}-\sqrt{b}$ *be given.*

$$\begin{array}{rl}
\text{Multiply by} & \underline{\sqrt{a}+\sqrt{b}} \\
\text{product} & a-b \quad \text{rational.}
\end{array}$$

Ex. 2.

Let $\sqrt[4]{5}+\sqrt[4]{3}$ *be propoſed.*

Multiply by $\sqrt[4]{5}-\sqrt[4]{3}$

1 product $\sqrt{5}-\sqrt{3}$

multiply by $\sqrt{5}+\sqrt{3}$

2 product $5-3=2$, rational.

Ex. 3.

Let there be given $\sqrt[8]{a}+\sqrt[8]{b}$.

Multiply by $\sqrt[8]{a}-\sqrt[8]{b}$

1 product $\sqrt[4]{a}-\sqrt[4]{b}$

multiply by $\sqrt[4]{a}+\sqrt[4]{b}$

2 product $\sqrt{a}-\sqrt{b}$

mult. by $\sqrt{a}+\sqrt{b}$

3 product $a-b$ rational.

Ex. 4.

Let $a+\sqrt[4]{b}$ *be given.*

Multiplier $a-\sqrt[4]{b}$

1 prod. $aa-\sqrt{b}$

mult. $aa+\sqrt{b}$

2 prod. a^4-b

Cor. *The number of operations is equal to the power of* 2 *in the index.*

PROBLEM XXXVI.

Any binomial ſurd being given, to find a multiplier which ſhall produce a rational product.

RULE.

If the ſurds have not the ſame index, reduce them to the ſame, (Prob. 21.)

Take the two quantities (throwing away the radical ſign or index); change the ſign of one of them. That done, involve theſe to the next inferior power denoted by the index of the root (Prob. 5. Rule 3), but leave out the unciæ or coefficients: then place the common radical ſign before each quantity, but after its ſign. And this will be your multiplier.

Shorter thus,

Binomial $\sqrt[n]{A} \pm \sqrt[n]{B}$.

Multiplier $\sqrt{A^{n-1}} \pm \sqrt[n]{A^{n-2}B} + \sqrt[n]{A^{n-3}B^2} \pm \sqrt[n]{A^{n-4}B^3} + \&c.$

The upper ſigns muſt be taken with the upper, and the lower with the lower; and the ſeries continued to n terms.

Ex. 1.

Let $\sqrt[3]{7} + \sqrt[3]{3}$ *be given.*

Multiplier $\sqrt[3]{7\times7} - \sqrt[3]{7\times3} + \sqrt[3]{3\times3}$

$$\begin{aligned} 7 &+ \sqrt[3]{7\times7\times3} \\ &- \sqrt[3]{7\times7\times3} - \sqrt[3]{7\times3\times3} \\ &\qquad\qquad\qquad + \sqrt[3]{7\times3\times3} + 3 \end{aligned}$$

product $7 + 3 = 10$, rational.

Ex. 2.

Let $a-\sqrt[3]{2}$ *be propoſed.* $=\sqrt[3]{a^3}-\sqrt[3]{2}$.

Multiplier $aa+a\sqrt[3]{2}+\sqrt[3]{2\times 2}$

product a^3-2.

Ex. 3.

Let $\sqrt[3]{a}+\sqrt[3]{b}$ *be propoſed.*

Mult. $\sqrt[3]{aa}-\sqrt[3]{ab}+\sqrt[3]{bb}$

product $a \qquad +b$

Ex. 4.

Let $\sqrt[4]{5}+\sqrt{3}$ *be given.*

reduced $\sqrt[4]{5}+\sqrt[4]{9}$, given.

Multiplier $\sqrt[4]{5^3}-\sqrt[4]{5^2\times 9}+\sqrt[4]{5\times 9^2}-\sqrt[4]{9^3}$

product $5 \qquad -9=-4$.

Or thus,

Surd $\sqrt[4]{9}+\sqrt[4]{5}$.

mult. $\sqrt[4]{9^3}-\sqrt[4]{9^2\times 5}+\sqrt[4]{9\times 5^2}-\sqrt[4]{5^3}$

product $9 \quad -5 \qquad =4$

Ex. 5.

Let $\sqrt[4]{a^3}-\sqrt[4]{b^3}$ *be given.*

Multiplier $\sqrt[4]{a^9}+\sqrt[4]{a^6b^3}+\sqrt[4]{a^3b^6}+\sqrt[4]{b^9}$.

Or thus,

Surd $\sqrt[4]{a^3} - \sqrt[4]{b^3}$.
mult. $\underline{aa\sqrt[4]{a} + a\sqrt[4]{aab^3} + b\sqrt[4]{a^3bb} + bb\sqrt[4]{b}}$
product $\underline{a^3 - b^3}$.

Ex. 6.

Let $\sqrt{a} - \sqrt[3]{b}$ *be propoſed.*
reduced to $\sqrt[6]{a^3} - \sqrt[6]{bb}$
put $x = a^3$, $y = bb$.
Surd $\sqrt[6]{x} - \sqrt[6]{y}$
mult. $\underline{\sqrt[6]{x^5} + \sqrt[6]{x^4y} + \sqrt[6]{x^3y^2} + \sqrt[6]{x^2y^3} + \sqrt[6]{xy^4} + \sqrt[6]{y^5}}$.
product $\underline{x - y = a^3 - bb.}$

PROBLEM XXXVII.

A fraction being given whoſe denominator is a compound ſurd; to reduce it to another whoſe denominator is rational.

RULE.

Find ſuch a multiplier (by Prob. 34, 35, or 36), as will make the denominator rational. By this multiply both numerator and denominator.

Ex. 1.

Let $\frac{3}{\sqrt{5} - \sqrt{2}}$ *be propoſed.*

Here $\frac{3 \times \overline{\sqrt{5} + \sqrt{2}}}{\overline{\sqrt{5} - \sqrt{2}} \times \overline{\sqrt{5} + \sqrt{2}}} =$

$\frac{3\sqrt{5} + 3\sqrt{2}}{5 - 2 = 3} = \sqrt{5} + \sqrt{2}$.

Ex. 2.

Let there be given $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}$.

Multiply both terms by $\sqrt{7}-\sqrt{3}$, the fraction becomes $\frac{\sqrt{42}-\sqrt{18}}{7-3=4}$.

Ex. 3.

Suppoſe $\frac{\sqrt{2}}{3-\sqrt{2}}$.

Multiply by $3+\sqrt{2}$, then $\frac{3\sqrt{2}+2}{9-2=7}$ is the fraction required.

Ex. 4.

Let $\frac{ab-b\sqrt{bc}}{a+\sqrt{bc}}$ *be propoſed.*

Multiply by $a-\sqrt{bc}$, then $\frac{aab-2ab\sqrt{bc}+b^2c}{aa-bc}$ is the fraction ſought.

Ex. 5.

Let $\frac{3\sqrt{a}+2\sqrt{b}}{5-\sqrt{3}}$ *be given.*

Multiply by $5+\sqrt{3}$; then $\frac{15\sqrt{a}+10\sqrt{b}+3\sqrt{3a}+2\sqrt{3b}}{25-3=22}$.

Ex. 6.

Suppoſe $\frac{10}{\sqrt[3]{7}-\sqrt[3]{5}}$.

Multiply by $\sqrt[3]{7^2}+\sqrt[3]{7\times5}+\sqrt[3]{5^2}$, and the fraction becomes $\frac{10\sqrt[3]{49}+10\sqrt[3]{35}+10\sqrt[3]{25}}{7-5}=$ $5\sqrt[3]{49}+5\sqrt[3]{35}+5\sqrt[3]{25}$.

Ex.

Ex. 7.

Let $\dfrac{\sqrt{ab}}{\sqrt[4]{5}+\sqrt[4]{3}}$.

Multiply by $\sqrt[4]{5^3}-\sqrt[4]{5^2\cdot 3}+\sqrt[4]{5\cdot 3^2}-\sqrt[4]{3^3}$,
And the fraction is

$$\frac{\sqrt[4]{125}-\sqrt[4]{75}+\sqrt[4]{45}-\sqrt[4]{27}}{5-3=2}\sqrt{ab}.$$

Or thus,

Multiply the terms of the fraction $\dfrac{\sqrt{ab}}{\sqrt[4]{5}+\sqrt[4]{3}}$ by $\sqrt[4]{5}-\sqrt[4]{3}$, and it becomes $\dfrac{\sqrt[4]{5}-\sqrt[4]{3}}{\sqrt{5}-\sqrt{3}}\sqrt{ab}$; again multiply the terms of the last fraction by $\sqrt{5}+\sqrt{3}$, and it becomes

$$\frac{5^{\frac{3}{4}}-5^{\frac{1}{2}}3+{}^{\frac{1}{4}}3^{\frac{1}{2}}5^{\frac{1}{4}}-3^{\frac{3}{4}}}{5-3=2}\sqrt{ab}.$$

Ex. 8.

Let $\dfrac{8}{\sqrt{3}+\sqrt{2}+1}$ *be the fraction.*

Multiply by $\sqrt{3}+\sqrt{2}-1$, and the fraction will be $\dfrac{8\sqrt{3}+8\sqrt{2}-8}{5+2\sqrt{6}-1}=\dfrac{4\sqrt{3}+4\sqrt{2}-4}{2+\sqrt{6}}$.
Again, multiply by $-2+\sqrt{6}$, and it becomes

$$\frac{-8\sqrt{3}-8\sqrt{2}+8+4\sqrt{18}+4\sqrt{12}-4\sqrt{6}}{6-4=2}$$

$$=4+2\sqrt{18}+2\sqrt{12}-2\sqrt{6}-4\sqrt{3}-4\sqrt{2}$$
$$=4+6\sqrt{2}+4\sqrt{3}-2\sqrt{6}-4\sqrt{3}-4\sqrt{2}$$
$$=4+2\sqrt{2}-2\sqrt{6}.$$

SECT. IV.

Several Methods of managing Equations.

AN *Equation* is the mutual comparing of two equal quantities, by the help of this character ($=$); the part on the left hand is called the *firſt ſide* of the equation; that on the right, the *ſecond ſide.* And the ſingle quantities are called *terms* of the equation.

An equation is either two ranks of quantities equal to one another, and ſeparated by this mark ($=$); or one rank equal to nothing. And they are to be conſidered either, as the laſt concluſion to which we come in the ſolution of a problem; or as the means whereby we come to it. In the firſt caſe, the equation is compoſed of only one unknown quantity mixed with known ones, and may be called the *final* equation. But thoſe of the laſt ſort involve ſeveral unknown quantities; and therefore they are to be ſo managed and reduced, that out of all the reſt there may emerge a new equation, with only one unknown quantity, which is that we ſeek. And this is to be made as ſimple as it can, in order to find the value of the unknown quantity.

An equation is named according to the dimenſion of the higheſt power of the unknown quantity in it. A *ſimple equation* is that which contains only the quantity itſelf; as $a = b - c$. A *quadratic equation*, is when the higheſt power is a ſquare, as $aa - ba = d$. A *cubic equation*, when the higheſt power is a cube, as $a^3 + ba^2 - ca = d$. A *fourth power* when the higheſt power is ſuch, as $a^4 - 3a^3 + a = d$, &c.

PRO-

PROBLEM XXXVIII.

To turn proportional quantities into equations; and equations into proportions.

In the ſolution of problems, it often happens, that we have ſeveral quantities in geometrical proportion, which are to be reduced into an equation; which will be done thus:

RULE.

Multiply the extremes together for one ſide of the equation, and the two means for the other ſide; or the ſquare of the mean, when there are but 3 terms.

On the contrary in a given equation, divide each ſide into two factors; and make the two factors of one ſide the two means; and the two factors of the other ſide, the extreams.

Ex. 1.

If $a : b :: c + f : d$. Then $ad = bc + bf$.

Ex. 2.

Let $a + b : a - b :: \frac{c}{d}\sqrt{aa - xx} : \frac{r}{s}$.

Then $\frac{ar + br}{s} = \frac{ca - cb}{d}\sqrt{aa - xx}$.

Ex. 3.

If $ad = bc + bf$. Then $a : b :: c + f : d$.

Ex. 4.

If $\frac{ar+br}{s} = \frac{ca-cb}{d}\sqrt{aa-xx}.$

Then $\frac{a+b}{s} : \frac{ca-cb}{d} :: \sqrt{aa-xx} : r$

or $a-b : a+b :: \frac{r}{s} : \frac{c}{d}\sqrt{aa-xx}.$

Ex. 5.

Let $bc+bd = da-cg.$

Then $1 : b :: c+d : da-cg.$

or $b : \sqrt{da-cg} :: \sqrt{da-cg} : c+d.$

PROBLEM XXXIX.

To reduce an equation.

When a queſtion is brought to an equation, the unknown quantities are generally mixed and entangled with the known ones; and therefore the equation muſt be ſo ordered that the unknown quantity may ſtand clear, on the firſt ſide of the equation; and the known ones on the ſecond ſide. Which is done thus:

1 RULE.

When any quantity is on both ſides the equation, throw it out of both.

Ex. 1.

If $3x+6b = 4c-d+9b.$

Throw out $6b$. Then $3x = 4c-d+3b.$

2 RULE.

When the known and unknown quantities are both on one ſide; tranſpoſe any of them to the contrary ſide, and change its ſign.

Ex. 2.

If $5x + 3b = rx + bd$.
Then $5x = rx + bd - 3b$.
And $5x - rx = bd - 3b$.

For to tranſpoſe a quantity with a contrary ſign, is the ſame thing as to add it, or elſe to ſubtract it from both ſides; therefore the quantities on each ſide, remain ſtill equal, by Axiom 1. and 2.

3 RULE.

If there be fractions in the equation, multiply both ſides by the denominators.

Ex. 3.

Suppoſe $\frac{aa}{b} + c - f = \frac{dx}{a}$.

Multiply by b, $aa + cb - fb = \frac{bdx}{a}$

multiply by a, $a^3 + bca - bfa = bdx$.

This proceſs is plain from Axiom 3.

4 RULE.

When any quantity is multiplied into both ſides of the equation, or into the higheſt term of the unknown quantity; divide the whole equation thereby.

Ex. 4.

If $7ba^3 + bcaa = bcda$.
Divide by ba, $7aa + ca = cd$.
divide by 7, $aa + \frac{ca}{7} = \frac{cd}{7}$.

The truth of this appears by Axiom 4.

5 RULE.

If the unknown quantity is affected with a ſurd; tranſpoſe the reſt of the terms; then involve each ſide according to the index of the ſurd.

Ex. 5.

If $\sqrt{aa - ba} + c = d$.

Then $\sqrt{aa - ba} = d - c$.

ſquared, $aa - ba = dd - 2dc + cc$.

This proceſs is plain from Axiom 5.

6 RULE.

When the ſide containing the unknown quantity is a pure power; or if being adfected, it has a rational root: then extract ſuch root on both ſides of the equation.

Ex. 6.

If $a^3 = b^3 - bbc$.

Cube root. $a = \sqrt[3]{b^3 - bbc}$.

Ex. 7.

If $xx + 6x + 9 = 20b$.

Square root $x + 3 = \pm\sqrt{20b}$.

and $x = \pm\sqrt{20b} - 3$.

Schol. All theſe rules are to be uſed promiſcuouſly, as one has occaſion for them, till the equation be duly cleared.

PROBLEM XL.

To explain the nature and origin of adfected equations.

1. Any adfected equation may be conſidered as made up of as many ſimple equations, as the di-

menſion of the higheſt power is. Suppoſe $x = a$, $x = b$, and $x = c$, &c. then $x - a = 0$, $x - b = 0$, $x - c = 0$. And if all theſe be multiplied together, then $\overline{x - a} \times \overline{x - b} \times \overline{x - c} = 0$; that is,

$$\begin{array}{llll} x^3 - a x^2 & + abx & - abc = 0, & \text{a cubic equation,} \\ - b & + ac & & \\ - c & + bc & & \end{array}$$

whoſe roots are a, b, c.

In like manner, $\overline{x - a} \times \overline{x - b} \times \overline{x - c} \times \overline{x - d} = 0$, produces a biquadratic equation,

$$\begin{array}{lllll} x^4 - a x^3 & + ab\, x^2 & - abc\, x & + abcd = 0, \\ - b & + ac & - abd & \\ - c & + bc & - acd & \\ - d & + da & - bcd & \\ & + db & & \\ & + dc & & \end{array}$$

whoſe roots are a, b, c, d.

Theſe two equations may be written or denoted thus, $x^3 - px^2 + qx - r = 0$, and $x^4 - px^3 + qx^2 - rx + s = 0$. And any ſuch equation being found in the ſolution of a problem; the buſineſs is then to reſolve it into its original compounding ſimple equations, and ſo to find the roots a, b, c, &c. For each of theſe ſimple equations gives one value of x, or one root. And if any one of theſe values of x be ſubſtituted in the equation inſtead of x, all the terms of the equation will vaniſh and be $= 0$. For ſince $\overline{x - a} \times \overline{x - b} \times \overline{x - c}$, &c. $= 0$. It is plain, when one of the factors $\overline{x - a}$ is $= 0$, the whole product will be $= 0$. And of conſequence there are three roots in the cubic equation, and four in the biquadratic; and in general there are as many roots, as is the dimenſion of the higheſt power in it, and no more.

2. If it happen that the roots a, b, c, &c. be equal to one another, then $\overline{x-a}^{3}$ will be $=0$; or $\overline{x-a}^{4}=0$, &c. and $x-a$ is had by evolution, ſince the given equation is generated by involution.

3. That there are no more roots than theſe is plain; for if you put any quantity, as f for x, which is equal to none of the roots a, b, c, &c. Then ſince neither $f-a$, $f-b$, nor $f-c$, &c. is 0, their product cannot vaniſh or be $=0$, but muſt be ſome real product; and therefore f is no root of the equation.

4. Since the ſquare root of a negative quantity is impoſſible; therefore if we have ſuch an equation as this, $xx+aa=0$, or $xx=-aa$, then $x=\pm\sqrt{-aa}$, which are two impoſſible roots of that equation. So that a quadratic equation has either two impoſſible roots or none. And therefore in any equation, there is always an even number of impoſſible roots; ſince each quadratic that goes to the compounding it, muſt have either two or none. Therefore no equation can have an odd number of impoſſible roots. Hence therefore the number of real roots in a cubic equation, will either be one or three; in a biquadratic, four, two, or none. In a fifth power, 5, 3 or 1; *&c.*

5. From the foregoing equations it is plain, that the coefficient of the firſt term (or that of the higheſt power) is 1. The coefficient of the ſecond term, (or next higheſt power), is the ſum of all the roots, a, b, c, &c. with their ſigns changed. The coefficient of the third term, the ſum of the products of every two of the roots. The coefficient of the fourth term, the ſum of the products of every three of them, with contrary ſigns, *&c.* The odd terms having always the ſame ſign, and the even terms a contrary one. And the

abſolute number is always the product of all the roots together.

6. Hence it follows, that when the ſum of all the negative roots is equal to the ſum of all the affirmative, the ſecond term vaniſhes, and the contrary. And if all the negative rectangles be equal to all the affirmative ones, the third term vaniſhes. And if all the negative ſolids be equal to all the affirmative ones, the fourth term vaniſhes, out of the equation; and ſo forward.

7. But the roots of equations may be either + or —, yet ſtill the ſame rules hold good. For let the ſign of any of them as c be changed into $-c$, that is, let $x+c=0$; then in the cubic equation the ſecond term will be $-a-b+c$; that is, the ſum of the roots with a contrary ſign; the third term will be $+ab-ac-bc$, that is, the ſum of the products of all the roots; and ſo of the reſt.

8. Hence alſo in every equation cleared of fractions and ſurds, each of the roots, each of the rectangles of any two of the roots, each of the ſolids under any three of them, each of the products of any four of the ſaid roots, *&c.* are all of them juſt diviſors of the laſt term or abſolute number. Therefore when no ſuch diviſor can be found, it is evident there is no root, no rectangle of roots, no ſolid of roots, *&c.* but what is ſurd. For in the cubic equation, a, b, c, and ab, ac, bc, are all of them diviſors of the laſt term abc: and ſo of higher powers.

9. In any equation, change the ſigns of all the terms but the firſt; then let the coefficients of the firſt, ſecond, third, *&c.* terms be 1, p, q, r, s, t, v, &c. reſpectively.

Then obſerving the ſigns, we ſhall have

$p =$ ſum of the roots, $a + b + c$ &c.

$pA + 2q =$ ſum of the ſquares of the roots $= a^2 + b^2$ &c.

$pB + qA + 3r =$ the ſum of their cubes $= a^3 + b^3$ &c.

$pC + qB + rA + 4s =$ the ſum of the biquadrates, *&c.*

Where A, B, C, *&c.* are the firſt, ſecond, third, *&c.* terms.

For $+p = a + b + c$ &c. $=$ A.

Alſo pA or $\overline{a + b + c}^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2cd = B - 2q$. Therefore $B = pA + 2q$, &c.

To go through the calculations of the reſt would be tedious, and of little uſe.

10. In equations of the third and fourth power, we find, when the roots are all affirmative, the ſigns are + and — alternately; ſo that there are as many changes of the ſines as is the index of the power, or as the number of roots. But if the roots are all negative, the ſigns are all + throughout, there being no changes of the ſigns. Whence in theſe caſes, there are as many affirmative roots, as changes of the ſigns in all the terms, from + to —, and from — to +. And the ſame rule holds in general, that is, there are as many affirmative roots in any equation as there are changes of the ſigns. But the equation is ſuppoſed to be compleat, that is to want no terms, and to have numeral coefficients. And likewiſe the number of negative roots is known thus; as often as two of the ſigns +, or two of the ſigns — ſtand next one another, ſo often there is a negative root. It would be needleſs to trouble the reader with the proof of theſe things; ſince it can only be done in particular caſes, and not in a general way.

And beſides when impoſſible roots happen to lie hid in the equation, they cauſe the rule to fail.

11. When the roots are all affirmative, the terms of the equation are alternately + and — through the equation; but when the roots are all negative, the ſigns are all +; and therefore, as by changing the ſigns of the roots, the ſigns of the alternate terms are changed; ſo on the contrary, changing the ſigns of the alternate terms, changes the ſigns of all the roots. And this holds in general, as will be evident by producing two equations from the ſame roots, with contrary ſigns.

12. Since any adfected equation, as $x^3 - px^2 + qx - r = 0$, is made up of ſimple equations, ſuch as $x - a = 0$, $x - b = 0$, &c. Thereſore if one root as a be known, the whole equation may be exactly divided by $x - a$; and ſo reduced to a lower dimenſion. Alſo when all the roots a, b, c are found out, then will the continual product of $x - a$, $x - b$, $x - c$, exactly produce the ſame equation. It is no wonder that an equation has ſeveral roots; becauſe in ſuch caſes, there are more ſolutions to a problem than one. So that in one caſe of it, x is $= a$, in another caſe $x = b$, in a third $x = c$, &c. and they are all comprehended in the general equation. And hence though there be ſeveral roots in an equation, yet only one of them will anſwer one caſe, or the particular queſtion propoſed.

12. That any root ſubſtituted for x in the given equation, will make the whole equation to vaniſh, by deſtroying all the terms, is proved thus. Let the equation be,

$$\begin{array}{llll} x^3 & - a\,x^2 & + ab\,x & - abc = 0. \\ & - b & + ac & \\ & - c & + bc & \end{array}$$

And let the roots be a, b, c, as before. Then ſubſtitute any one, as a, inſtead of x, and the equation will become

$$\begin{array}{lll} a^3 - a^3 & + baa - abc = 0, \\ \quad - baa & + caa \\ \quad - caa & + abc \end{array}$$

Where the terms manifeſtly deſtroy one another. And the ſame will happen, by ſubſtituting b or c, for x.

13. If the laſt term of an equation vaniſhes (as abc, Art 12), then one root will be 0; for then the whole equation may be divided by the unknown quantity x or $x - 0$. If the two laſt terms vaniſh ($abx + acx + bcx$, and $- abc$), then two roots are $= 0$; if the three laſt terms vaniſh, then three roots will be 0; *&c.*

And on the contrary, if one, two, or three roots, *&c.* be $= 0$, the laſt term, the two laſt, or the three laſt terms, *&c.* will vaniſh out of the equation, and the remaining part of the equation will contain the reſt of the roots. Thus in the equation, Art. 12. if the roots b, c be $= 0$; there remains only $x^3 - \overline{a + b + c} \times x^2 = 0$, or $x - a = 0$, an equation containing the remaining root a.

14. And in any power of a binomial, if each term be multiplied by the index of the unknown quantity therein; it will thereby be reduced to the next inferior power. To prove this, we muſt obſerve, that the coefficients of a binomial, are the very ſame, whether you reckon forward from the beginning, or backward from the end; that is, the firſt and laſt are the ſame; the ſecond and laſt but one; the third and laſt but two, *&c.* For the coefficients of any power of $x + b$, are the ſame as of $b + x$. In the quadratic $xx + 2bx + bb$,

the coefficients are 1, 2, 1. In the cubic $x^3+3x^2b+3xb^2+b^3$, they are 1, 3, 3, 1. In the fourth power they are 1, 4, 6, 4, 1. In the fifth power, 1, 5, 10, 10, 5, 1; and ſo on.

Therefore, let any power of $x+b$ be denoted thus, $x^n+nx^{n-1}b+\frac{n.\overline{n-1}}{2}x^{n-2}bb +$
$\frac{n.\overline{n-1}\ \overline{n-2}}{2.3}x^{n-3}b^3$ &c. . . . . . . +
$\frac{n.\overline{n-1}.\overline{n-2}}{2.3}x^3b^{n-3}+\frac{n.\overline{n-1}}{2}x^2b^{n-2} +$
$nxb^{n-1}+b^n$; n being the index of the power, and let m be that of the next inferior power, or $m=n-1$. Now let each term be multiplied by the index of x in each term; that is, by n, $n-1$, $n-2$, &c. and we ſhall have

$nx^n+n.\overline{n-1}.x^{n-1}b+\frac{n.\overline{n-1}.\overline{n-2}}{2}x^{n-2}bb +$
$\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3}x^{n-3}b^3$ &c. . . . . +
$\frac{n.\overline{n-1}.\overline{n-2}}{2}x^3b^{n-3}+n.\overline{n-1}.x^2b^{n-2} +$
$nxb^{n-1}+0$. And dividing all by nx, it becomes
$x^{n-1}+\overline{n-1}.x^{n-2}b+\frac{\overline{n-1}.\overline{n-2}}{2}x^{n-3}bb +$
$\frac{\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3}x^{n-4}b^3$ &c. . . . . +
$\frac{\overline{n-1}.\overline{n-2}}{2}x^2b^{n-3}+\overline{n-1}.xb^{n-2}+b^{n-1}$; that is, reſtoring m, $x^m+mx^{m-1}b+\frac{m.\overline{m-1}}{2}x^{m-2}bb +$
$\frac{m.\overline{m-1}.\overline{m-2}}{2.3}\times x^{m-3}b^3$ &c. . . . . $+\frac{m.\overline{m-1}}{2}$
$x^2b^{m-2}+mxb^{m-1}+b^m$, which is manifeſtly the m^{th} power of $x+b$.

15. Alſo if the equation reſulting from the laſt operation be taken, and its ſeveral terms again multiplied by the index of x in each term; it will be reduced to the next power below that, and ſo on for more operations. And therefore after each operation one root will be deſtroyed; or ſo many roots will be deſtroyed as there are operations, and the reſt will remain.

16. And further: If there be ſeveral equal roots of one ſort, and alſo ſeveral equal ones of another ſort, in any equation. And if the terms of that equation be multiplied by the ſeveral indexes of the unknown quantity in each term; an equation will ariſe wherein one of the equal roots of each ſort will be deſtroyed. And in general, whatever roots there be in any equation, if the terms be reſpectively multiplied by the indexes of the unknown quantity therein, an equation will come out wherein one root of every ſort will be deſtroyed, whether there be equal roots, or all different. But theſe things being of little conſequence, I ſhall not detain the reader any longer about them.

17. As impoſſible roots are ſuch as are produced from the ſquare roots of negative quantities: ſo impoſſible equations are thoſe produced from impoſſible roots; as this equation $a^4 - 4a^3 + aa + 10a + 22 = 0$, which is produced from theſe two, $aa + 2a + 2 = 0$, and $aa - 6a + 11 = 0$; the former produced from $a + 1 + \sqrt{-1}$, and $a + 1 - \sqrt{-1}$; and the latter from $a - 3 + \sqrt{-2}$, and $a - 3 - \sqrt{-2}$. Theſe ſort of equations have roots that are barely impoſſible.

Likewiſe, there are equations that are doubly impoſſible, or impoſſible equations of the ſecond degree. And theſe are produced from equations involving two degrees of impoſſibility, as this $a^4 + 4a^3 + 8aa + 8a + 5 = 0$, which is produced from the equations, $aa + 2a + 2 + \sqrt{-1} = 0$, and $aa+2a+2-\sqrt{-1}=0$. Such as theſe cannot be reduced into rational quadratics, as the others may.

PROBLEM XLI.

To increaſe or diminiſh the roots of an equation, by any given quantity.

RULE.

For the unknown letter ſubſtitute a new letter, − the given increment, or + the given decrement. And ſubſtitute the powers thereof, in the equation, inſtead of the powers of the unknown letter.

Ex. 1.

Let $x^3 - px^2 + qx - r = 0$, *be given; and let the roots be leſſened by the quantity* e.

Suppoſe $y = x - e$, or $x = y + e$. Then

$$\left.\begin{array}{rcrrrr} x^3 & = & y^3 & + 3ey^2 & + 3e^2y & + e^3 \\ -px^2 & = & & -py^2 & -2pey & -pe^2 \\ +qx & = & & & +qy & +qe \\ -r & = & & & & -r \end{array}\right\} = 0, \text{ which}$$

is the equation required.

Ex. 2.

Increaſe the roots by 4, *of this equation* $a^3 + a^2 - 10a + 8 = 0$.

Suppoſe $a + 4 = e$, or $a = e - 4$.

$$\begin{array}{lrcrrrr} \text{Then} & a^3 & = & e^3 & -12e^2 & +48e & -64 \\ & +a^2 & = & & ee & -8e & +16 \\ & -10a & = & & & -10e & +40 \\ & +8 & = & & & & +8 \\ \hline & & & e^3 & -11ee & +30e & * \end{array}$$

$e^3 - 11ee + 30e \; * = 0$, the equation required; reduced, $e^2 - 11e + 30 = 0$, a quadratic.

Cor. 1. *The laſt term of the tranſformed equation, is the very ſame as the equation given, having e in the place of r* (in Ex. 1.)

Cor. 2. *When the laſt term vaniſhes, the number aſſumed* (— 4, Ex. 2.) *is one of the roots in the equation propoſed.*

Schol. By this rule, all the roots of an equation may be made affirmative; by increaſing them by a proper quantity.

PROBLEM XLII.

To multiply or divide the roots of an equation, by a given number or quantity.

RULE.

Aſſume a new letter; and divide or multiply it by the given number; and ſubſtitute its powers in the equation, inſtead of the unknown quantity.

Ex. 1.

Multiply by 3, *this equation* $y^3 - \frac{4}{3}y - \frac{146}{27} = 0$.

Suppoſe $y = \frac{1}{3}z$, then ſubſtituting $\frac{1}{3}z$ for y, we have $\frac{z^3}{27} - \frac{4}{9}z - \frac{146}{27} = 0$, or reduced $z^3 - 12z - 146 = 0$.

Ex. 2.

Divide by $\surd 3$, *the equation* $x^3 - 2x + \surd 3 = 0$.

Let $x = y\surd 3$, which put for x, we have $3y^3\surd 3 - 2y\surd 3 + \surd 3 = 0$, or $3y^3 - 2y + 1 = 0$.

Cor. *By this rule, fractions or ſurds may be taken out of an equation; by dividing the new letter by the common denominator; or by multiplying the new letter by the ſurd quantity.*

PROBLEM XLIII.

To change the roots of an equation into their reciprocals.

RULE.

In the given equation, inſtead of the root, ſubſtitute a unit divided by ſome other letter.

Example.

Let $3y^3 - 2y + 1 = 0$, *be given.*

Put $y = \frac{1}{z}$, then $\frac{3}{z^3} - \frac{2}{z} + 1 = 0$.

reduced $3 - 2z^2 + z^3 = 0$.

or $z^3 - 2z^2 + 3 = 0$.

Schol. By this rule the greateſt root is changed into the leaſt, and the leaſt into the greateſt, *&c.*

PROBLEM. XLIV.

To compleat a deficient equation.

An equation is *compleat*, when it has all its terms, or thoſe containing all the powers of the unknown quantity; and *deficient*, when any power is wanting.

RULE.

Increaſe or diminiſh the roots of the equation, by ſome given quantity (by Prob. 41).

Example.

Suppoſe $a^3 + 2a - 5 = 0$, *deficient.*

Let $e + 1 = a$, then

$$\begin{array}{rcr} a^3 & = & e^3 + 3ee + 3e + 1 \\ +2a & = & +2e + 2 \\ -5 & = & -5 \\ \hline & & e^3 + 3ee + 5e - 2 = 0, \end{array}$$

compleat.

Schol. An equation may be rendered compleat, by multiplying by the ſame letter with ſome quantity added, as $a+1$; but then it raiſes the equation a degree higher.

PROBLEM XLV.

To depreſs an equation to a lower dimenſion; one of its roots being given.

1 RULE.

Put the equation $=0$, and divide it by the unknown quantity — the root given.

Example.

Given $a^3+a^2-10a+8=0$, *one root* $a=-4$.

$$\begin{array}{rl} a+4=0)\ & a^3+a^2-10a+8=0\ (aa-3a+2=0 \text{ the equation req.}\\ & a^3+4a^2 \\ \hline & -3a^2-10a \\ & -3a^2-12a \\ \hline & \qquad +2a+8 \\ & \qquad +2a+8 \\ \hline & \qquad\quad 0 \\ \hline \end{array}$$

2 RULE.

Put a new letter added to that root, equal to the unknown quantity; and ſubſtitute that and its powers in the equation.

Example.

Let $a^3+a^2-10a+8=0$, *be given, and* $a=-4$. Put $a=e-4$. Then

$$\begin{array}{rcr} +a^3 &=& e^3-12e^2+48e-64\\ +a^2 &=& +ee-8e+16\\ -10a &=& -10e+40\\ +8 &=& 8\\ \hline 0 &=& e^3-11e^2+30e+0 \end{array}$$

reduced $e^2-11e+30=0$

PROBLEM XLVI.

To find how many roots are affirmative, and how many negative, in a given equation.

RULE.

Range the terms of the equation according to the dimenſions of the unknown quantity. And if the equation is not compleat, make it ſo by Prob. 44.

Then obſerve how often + follows —, or — follows +, that is, how many changes of the ſigns there are; and there are ſo many affirmative roots in the equation.

Alſo, as often as two like ſigns ſtand together, ſo often there is a negative root.

Ex. 1.

Given $x^4 - x^3 - 19xx + 49x - 30 = 0$.

Here the ſigns are + — — + —, and there are three changes; from the firſt to the ſecond, from the third to the fourth, and from the fourth to the fifth term: therefore there are three affirmative roots. Alſo, in the ſecond and third terms, two negatives ſtand together, and in none elſe, conſequently there is one negative root.

Ex. 2.

Suppoſe $x^4 + 5x^3 - 7x^2 - 29x + 30 = 0$.

The ſigns are + + — — +
roots neg. af. neg. af.

So there are two affirmative, and two negative roots.

Ex.

Ex. 3.

Let the equation be $a^3 - 7a + 6 = 0$.

This equation being defective is to be compleated.

$$\begin{array}{rl} & a^3 \; * \; - 7a + 6 = 0. \\ \text{mult. by} & a + 1 = 0. \\ \hline & a^4 \;\; * \; - 7a^2 + 6a \\ & + a^3 \;\; * \; - 7a + 6. \\ \hline & a^4 + a^3 - 7a^2 - a + 6 = 0. \end{array}$$

So there are two affirmative, and two negative roots in this laſt equation, and one of the negative roots being -1, (by the multiplication of $a + 1 = 0$,) therefore, the given equation contains two affirmative roots, and one negative.

The reaſon of this rule appears from Art. 10. Prob. 40.

SCHOLIUM.

This rule does not hold good, if there be impoſſible roots in the equation; except ſo far as theſe impoſſible roots may be taken for ambiguous ones, that is, for either affirmative or negative roots. As in the equation $x^3 - 6x^2 + 13x - 10 = 0$, which by this rule gives three affirmative roots, but in reality it has but one root, which is 2, the reſt are imaginary.

There are alſo ſome rules whereby to judge how many impoſſible roots are in an equation, but they are ſo very tedious, and of ſo little uſe, that I ſhall not trouble the reader with them. See *Newton*'s *Univerſal Arithmetic*, p. 197.

PROBLEM XLVII.

To change the affirmative roots into negatives, and the negatives into affirmatives.

RULE.

Place cyphers for the deficient terms, if there be any; then change the ſigns of all the even terms, that is, of the ſecond, fourth, ſixth, *&c.* terms of the equation.

Ex. 1.

Given $x^3 + 8x + 24 = 0.$

That is, $x^3 + 0 + 8x + 24 = 0.$

transformed $x^3 - 0 + 8x - 24 = 0.$

In the given equation $x = -2$, in the transformed equation $x = +2$.

Ex. 2.

Suppoſe $+x^4 - 4x^3 - 19x^2 + 106x - 120 = 0.$

transformed $+x^4 + 4x^3 - 19x^2 - 106x - 120 = 0.$

In the former equation the roots are 2, 3, 4 and -5; and in the latter 5, -2, -3, and -4.

The reaſon of this proceſs is plain from Art. 11. Prob. 40. and may be demonſtrated thus. In the given equation, we have $+x$ for the root. Now ſuppoſe $-x$ to be a root. Let this be ſubſtituted in the given equation, and it produces $-x^3 - 8x + 24 = 0$, that is, $x^3 + 8x - 24 = 0$, as in Exam. 1. And $x^4 + 4x^3 - 19x^2 - 106x - 120 = 0$, as in Exam. 2. For it is plain, all the odd powers of x will now be negative, which before were affirmative, the reſt remaining as before. Whence the ſigns of all the odd powers will be changed, according to the rule.

SECT. V.

Ranging the terms; working by general forms; ſubſtitution and reſtitution; taking away any term of an equation; extermination of unknown quantities; the deſignation of quantities by letters; regiſtering the ſteps.

PROBLEM XLVIII.

To range the terms of an equation, or diſpoſe of them in the beſt manner for any operation.

RULE.

THIS is done by placing theſe terms foremoſt that contain the higheſt power of the unknown quantity; and in the following places, thoſe of leſs dimenſions; ſo that the powers in the ſeveral terms may continually decreaſe from the higheſt, according to the ſeries of the natural numbers. But in many caſes, the contrary method is to be followed, and the loweſt power taken firſt.

Ex. 1.

Let $az^3 + z^4 - bz^3 - b^4 + ab^3 = 0$.

Place it thus, $$\begin{array}{lllll} z^4 & +az^3 & * & * & +ab^3 = 0. \\ & -b & & & -b^4 \end{array}$$

Ex. 2.

Suppoſe $x^4 + ax^3 + bx^2 - bx^3 + cx = dx - ab^3 + b^4$;

ranged $$\begin{array}{llllll} x^4 & +ax^3 & +bx^2 & +cx & +ab^3 & = 0. \\ & -b & & -d & -b^4 & \end{array}$$

PRO.

PROBLEM XLIX.

To work by a general form.

RULE.

Write down each letter or quantity in the general form, and after it (with the ſign =), each letter it repreſents in that particular caſe; which will give ſeveral equations.

Then caſt your eye over the general form, and obſerve the general quantities therein, and look for them on the firſt ſide of the equations; and what you find them equal to, on the right hand, write down, inſtead of them, each one by one, till you have gone through the general form; and you will have the ſolution.

When the quantities are many, it will be the beſt way to write down the general form firſt, and the particular one under it, each quantity under its correſpondent; then it will appear by inſpection what letters to ſubſtitute.

Ex. 1.

To involve $aa - xx$ *to the* 5*th power.*

This is to be done by the general form in Cor. 1. Prob. 5. therefore we have

$$a = aa$$
$$e = -xx$$
$$n = 5,$$

Whence $\overline{a+e}^{\,n} = \overline{aa - xx}^{\,5} = \overline{aa}^{\,5} + 5 \times \overline{aa}^{\,4} \times -xx + 5 \times \frac{5-1}{2} \times \overline{aa}^{\,3} \times x^4 + 5 \times \frac{5-1}{2} \times \frac{5-2}{3}\, \overline{aa}^{\,2} \times -x^6 + 5 \times \frac{5-1}{2} \times \frac{5-2}{3} \times \frac{5-3}{4}\, aa \times x^8 + 5 \times \frac{5-1}{2} \times \frac{5-2}{3} \times \frac{5-3}{4} \times \frac{5-4}{5} \times -x^{10} = a^{10} - 5a^8x^2 + 10a^6x^4 - 10a^4x^6 + 5aax^8 - x^{10}$, the power required.

Ex.

Ex. 2.

Extract the ſquare root of $28 - \sqrt{300}$.

This is to be done by the form in 1 Rule, Prob. 32.

Here $A = 28$, $B = \sqrt{300}$, $D = \sqrt{\overline{784 - 300}} = 22$; $\sqrt{\frac{A+D}{2}} = \sqrt{\frac{28+22}{2}} = 5$, $\sqrt{\frac{A-D}{2}} = \sqrt{\frac{28-22}{2}} = \sqrt{3}$. Therefore $\sqrt{\overline{A-B}} = 5 - \sqrt{3}$, the root required.

Ex. 3.

To find a quantity, by which if $\sqrt[5]{2} - \sqrt[5]{6}$ *be multiplied, the product will be rational.*

This is to be done by Prob. 36.

Here $n = 5$, $A = 2$, $B = 6$. And the multiplier $\sqrt[5]{16} + \sqrt[5]{8 \times 6} + \sqrt[5]{4 \times 36} + \sqrt[5]{2 \times 216} + \sqrt[5]{1296}$.

$$
\begin{array}{ll}
\text{mult.} & \sqrt[5]{16} + \sqrt[5]{8\times6} + \sqrt[5]{4\times36} + \sqrt[5]{2\times216} + \sqrt[5]{1296} \\
\text{by} & \sqrt[5]{2} - \sqrt[5]{6} \\
\hline
 & \sqrt[5]{32} + \sqrt[5]{96} + \sqrt[5]{8\times36} + \sqrt[5]{4\times216} + \sqrt[5]{2592} \\
 & \quad - \sqrt[5]{96} - \sqrt[5]{8\times36} - \sqrt[5]{4\times216} - \sqrt[5]{2592} - \sqrt[5]{7776} \\
\hline
 & 2 - 6 = -4. \quad \text{product.}
\end{array}
$$

PROBLEM L.

To ſhorten the work by ſubſtitution and reſtitution.

In any operation, when the quantities grow very numerous, or very much compounded, it will make the work very tedious; and therefore it ought to be made ſhorter as follows.

RULE.

RULE.

Aſſume a new letter to repreſent or ſtand for any number of given quantities; and likewiſe ſome different letter to ſtand for the coefficient of any power of the unknown quantity; do ſo for as many of the coefficients as are compounded. Likewiſe, put letters for the numbers concerned; then work with theſe inſtead of the original quantities, which will make the work eaſier. And this is called *Subſtitution.*

When the operation is over, each number or compound quantity muſt be reſtored again inſtead of its letter; and this is called *Reſtitution.*

Ex. 1.

Let $aa + ba - ca + da = dc$.

Put $s = b - c + d$. Then the equation becomes

$$aa + sa = dc.$$

Ex. 2.

Let $\overline{a - 2x} \times \sqrt{aa - xx} = \dfrac{bx}{cx^2 - dx^2 + cx}$.

Put $c - d = p$. Then

$$\overline{a - 2x}\sqrt{aa - xx} = \frac{bx}{pxx + cx}.$$

multiply by $pxx + cx$. Then

$$\overline{apxx - 2px^3 + acx - 2cxx} \times \sqrt{aa - xx} = bx.$$

Put $ap - 2c = q$. Then

$$\overline{qxx - 2px^3 + acx} \times \sqrt{aa - xx} = bx.$$

or $\overline{acx + qxx - 2px^3} \times \sqrt{aa - xx} = bx$.

ſquared $\overline{acx + qxx - 2px}^2 \times \overline{aa - xx} = bbxx$, &c. where the values of p, q, may be reſtored.

PROBLEM LI.

To take away the ſecond term of an equation.

RULE.

Divide the coefficient of the ſecond term by the index of the higheſt power; annex the quotient, with its ſign changed, to ſome new letter, which ſubſtitute for the root, in the given equation.

Ex. 1.

Suppoſe $a^3 + aa - 10a + 8 = 0$.

Put $e - \frac{1}{3} = a$. Then

$$\begin{array}{rl}
a^3 = & e^3 - e^2 + \frac{1}{3} + \frac{1}{27} \\
+ aa = & \quad + ee - \frac{2}{3} - \frac{1}{9} \\
-10a = & \qquad - 10e + \frac{10}{3} \\
+ 8 = & \qquad\qquad + 8 \\
\hline
0 = & e^3 \; * \; - 10\frac{1}{3}e + 11\frac{11}{27}
\end{array}$$

, the equation required.

Ex. 2.

Let $y^4 - 8ay^3 + a^4 = 0$, *be given.*

Let $y = x + \frac{8a}{4} = x + 2a$. Then

$$\begin{array}{rl}
y^4 = & x^4 + 8ax^3 + 24a^2x^2 + 32a^3x + 16a^4 \\
- 8ay^3 = & \quad - 8ax^3 - 48a^2x^2 - 96a^3x - 64a^4 \\
+ a^4 = & \qquad\qquad\qquad\qquad + a^4 \\
\hline
0 = & z^4 \; * \; - 24a^2x^2 - 64a^3x - 47a^4 = 0.
\end{array}$$

Schol. Hence by this and the 43d problem, an equation may be found, which wants the laſt term but one. For if the ſecond term be taken away by this problem, and the equation transformed by Prob. 43, you will have the equation required.

PROBLEM LII.

To take away any term out of an equation.

RULE.

Take a new letter for the root, to which add an unknown quantity; and ſubſtitute this ſum and the powers thereof, into the given equation. Then any term put equal to nothing, will determine the value of that aſſumed unknown quantity.

Ex. 1.

Suppoſe $x^4 - 3x^3 + 3x^2 - 5x - 2 = 0$.

Put $y + e = x$. Then

$$\left.\begin{array}{rl} x^4 & = y^4 + 4y^3e + 6yyee + 4ye^3 + e^4 \\ -3x^3 & = \quad -3y^3 - 9yye - 9ye^2 - 3e^3 \\ +3x^2 & = \qquad\qquad +3yy + 6ye + 3e^2 \\ -5x & = \qquad\qquad\qquad -5y - 5e \\ +2 & = \qquad\qquad\qquad\qquad +2 \end{array}\right\} = 0.$$

Then, if the ſecond term is to be taken away, make $4y^3e - 3y^3 = 0$, or $4e = 3$; therefore $e = \frac{3}{4}$.

Ex. 2.

The ſame ſuppoſed; to take away the third term.

Here we ſhall have $6y^2ee - 9y^2e + 3yy = 0$; reduced, $2ee - 3e + 1 = 0$, the reſolving of which quadratic equation gives the value of e. Then $y + e$ gives the value of x, ſo that the third term may vaniſh.

Ex. 3.

The ſame thing ſtill ſuppoſed; to take away the fourth or fifth term.

For the fourth term, $4e^3 - 9e^2 + 6e - 5 = 0$, a cubic equation whoſe root is e; and $y + e = x$, makes the fourth term vaniſh.

For the fifth term, $e^4 - 3e^3 + 3e^2 - 5e + 2 = 0$, a fourth power whoſe root is e. Then $y + e = x$, which ſubſtituted in the equation, makes the laſt term vaniſh.

Cor. 1. *Hence the third, fourth, fifth,* &c. *term, may be taken out of the equation; by reſolving a quadratic, cubic, fourth power,* &c. *equation.*

Cor. 2. *Hence if the laſt term of an equation (as* $e^4 - 3e^3 + 3ee - 5e + 2$*) be* $= 0$, *then one root* (x) *is* $= 0$; *for then* $x - 0$, *or* x *will divide the equation. If two of the laſt terms be* $= 0$, *two values of the root will be* $= 0$, *and ſo on. But if the laſt term does not vaniſh, there is no root* $= 0$.

Schol. After the ſame rule any term may be made equal to any given quantity; by putting the ſaid term equal to that quantity.

PROBLEM LIII.

To exterminate a ſingle letter, or a quantity of one dimenſion, out of ſeveral equations.

1 RULE.

Seek the value of the quantity to be expelled, in two equations; and put theſe values equal to one another.

Ex. 1.

$$\left.\begin{array}{l}\textit{Let } a+x=b+y\\ \textit{and } 2x+y=3b\end{array}\right\}\ \textit{to exterminate } y.$$

By tranſpoſing b, $a+x-b=y$, and by tranſpoſing $2x$, $y=3b-2x$. Therefore $a+x-b=3b-2x$. And by reduction $3x=4b-a$, and $x=\frac{4b-a}{3}$.

Ex. 2.

$$\left.\begin{array}{l}\textit{Let } ax-2by=ab\\ \textit{and } xy=bb,\end{array}\right\}\ \textit{to exterminate } y.$$

Here $ax-ab=2by$, and $y=\frac{ax-ab}{2b}$. Alſo $y=\frac{bb}{x}$; therefore $\frac{ax-ab}{2b}=\frac{bb}{x}$, and reducing $xx-bx=\frac{2b^3}{a}$:

2 RULE.

Find, by reduction, the value of one unknown quantity, in one equation; and ſubſtitute that value for it, in all the other equations. Proceed thus, with another unknown quantity, &c.

Ex. 3.

$$\left.\begin{array}{l}\textit{Let } a+x=2b-y\\ \textit{and } 3ax-yx=d\end{array}\right\}\ \textit{to expell } y.$$

By the firſt equation $y=2b-a-x$, put this value in the ſecond equation; then

$3ax-x\times\overline{2b-a-x}=d$, that is, $3ax-2bx+ax+xx=d$, or $4ax-2bx+xx=d$.

Ex. 4.

Suppoſe $\left.\begin{array}{r} x+y+z=a \\ 3y=x+2z \\ az=xy \end{array}\right\}$ *to expunge z and y.*

By the firſt equation, $z=a-x-y$,
By the ſecond equation, $3y=x+2a-2x-2y$,
By the third, $a\times\overline{a-x-y}=xy$,
or $aa-ax-ay=xy$.

The former reduced, $5y=2a-x$.
and ſince $aa-ax-ay=xy$.

From theſe to expunge y.

By the former $y=\dfrac{2a-x}{5}$, and by the latter $aa-ax=ay+xy$, and $y=\dfrac{aa-ax}{a+x}$. Therefore $\dfrac{2a-x}{5}=\dfrac{aa-ax}{a+x}$, in which equation there is only one unknown quantity x.

Cor. 1. *By each given equation, one unknown quantity may be taken away. And conſequently when there are as many equations as unknown quantities, they may be all taken away but one.*

Cor. 2. *If there be more unknown quantities than equations, there will remain in the laſt equation more unknown quantities by* 1, *than that exceſs amounts to.*

PROBLEM LIV.

To exterminate an unknown quantity of ſeveral dimenſions.

1 RULE.

Find the value of its greateſt power in two equations; then if they are not the ſame, multiply the leſſer power, ſo that it may become equal to the greater. Then put theſe values equal to each other, and there will come out a new equation, with a leſs power of the unknown quantity. And by repeating this operation, the quantity will at laſt be taken away.

Ex. 1.

$$\left.\begin{array}{l}\text{Let } aee + be + c = 0 \\ \text{and } fee + ge + h = 0\end{array}\right\}\ \textit{to expunge } e.$$

By tranſpoſing and dividing, $-ee = \dfrac{be + c}{a}$, and $-ee = \dfrac{ge + h}{f}$. Therefore $\dfrac{be + c}{a} = \dfrac{ge + h}{f}$. And multiplying, $bef + cf = age + ah$, and by tranſpoſing, $bfe - age = ah - cf$, and dividing, $e = \dfrac{ah - cf}{bf - ag}$. And multiplying by $-e$, $-ee = \dfrac{-ahe + cfe}{bf - ag}$. Whence $\dfrac{be + c}{a} = \dfrac{-ahe + cfe}{bf - ag}$. And multiplying alternately, $bbfe + bcf - abge - agc = acfe - aahe$. And tranſpoſing and dividing, $e = \dfrac{agc - bcf}{bbf - abg - acf + aah}$. Therefore $\dfrac{ah - cf}{bf - ag} = \dfrac{agc - bcf}{bbf - abg - acf + aah}$. Then multiplying and reducing,

$$\begin{array}{lll} hhaa & + cgga & + bbfh = 0. \\ & - 2cfh & - bgfc \\ & - bgh & + ccff \end{array}$$

2 RULE.

For two quadratic equations.

$$\left.\begin{array}{r} ax^2 + bx + c = 0, \\ \text{and } fx^2 + gx + h = 0,\end{array}\right\}\ \textit{to exterminate } x.$$

Here a, b, c, f, g, h, are either given quantities, or compoſed of given quantities, and ſome other unknown quantity y. Thus

make $bf-ag=A$, $bh-cg=B$, and $cf-ah=D$, then $AB+DD=0$.

To prove this rule, we have $-x^2=\frac{bx+c}{a}=\frac{gx+h}{f}$, which reduced is $\overline{bf-ag}\times x+cf-ah=0$; that is, $Ax+D=0$. Whence $Ax^2+Dx=0$; therefore $-xx=\frac{Dx}{A}=\frac{bx+c}{a}$, which reduced is $x=\frac{cA}{aD-bA}$. In like manner $-xx=\frac{Dx}{A}=\frac{gx+h}{f}$, which reduced is $x=\frac{hA}{fD-gA}$. Whence $\frac{cA}{aD-bA}=\frac{hA}{fD-gA}$. And this reduced is $\overline{cf-ah}\times D+\overline{bh-cg}\times A=0$, that is $AB+DD=0$.

The Newtonian *Rule is,*

$$ah\times\overline{ah-bg-2cf}+bf\times\overline{bh-cg}+c\times\overline{agg+cff}=0.$$

3 RULE.

For a cubic and a quadratic equation.

$$ax^3+bx^2+cx+d=0,$$
and $$fx^2+gx+h=0.$$

Make $\overline{fc-ah}=D$, $\overline{fb-ag}=A$.

Then $\overline{fD-gA}\times\overline{hD-fdg}+\overline{dff-hA}\vert^2=0$.

For, multiplying the firſt equation by f, and the ſecond by ax, and ſubtracting one from the other, we have $\overline{bf-ag}\times x^2+\overline{fc-ah}\times x+fd=0$; and ſince $fx^2+gx+h=0$, theſe two equations come under the laſt rule, making $a = \overline{bf - ag}$, $b = \overline{fc - ah}$, $c = fd$. And $A = f \times \overline{fc - ah} - g \times \overline{bf - ag}$, $B = h \times \overline{fc - ah} - fdg$, $D = ffd - h \times \overline{bf - ag}$. Whence by that rule, $\overline{f \times \overline{fc - ah} - g \times \overline{bf - ag}} \times \overline{h \times \overline{fc - ah} - fdg} + \overline{ffd - h \times \overline{bf - ag}}|^2 = 0$, that is, according to the preſent deſignation of the letters A, B, C; $\overline{fD - gA} \times \overline{hD - fdg} + \overline{ffd - hA}^2 = 0$.

The Newtonian *Rule is,*

$$\left.\begin{matrix} ahh \times \overline{ah - bg - 2cf} + bfh \times \overline{bh - cg - 2df} \\ + \overline{ch - dg} \times \overline{agg + cff} + df \times \overline{3agh + bgg + dff} \end{matrix}\right\} = 0.$$

4 RULE.

For a quadratic and a fourth power.

$$ax^4 + bx^3 + cx^2 + dx + e = 0,$$

and $$fx^2 + gx + h = 0.$$

Make $A = bf - ag$, $D = cf - ah$.

Then $df^3 \overline{- gfD + \overline{gg - fh}} \times A \times \overline{dhff - egff - hhA}$
$\overline{+ ef^3 + ghA - fhD}|^2 = 0$.

For, multiply the firſt equation by f, and the latter by axx, their difference will be $\overline{bf - ag} \times x^3 + \overline{cf - ah} \times x^2 + dfx + ef = 0$. Or $Ax^3 + Dx^2 + dfx + ef = 0$. And ſince $fx^2 + gx + h = 0$. Therefore theſe two equations come under the laſt rule; in which writing A for a, D for b, df for c, ef for d; and laſtly $fD - gA$ inſtead of A, and $ffd - hA$ for D, you will get the rule, as above.

The Newtonian *Rule is,*

$$\left.\begin{matrix} ah^3 \times \overline{ah - bg - 2cf} + bfhh \times \overline{bh - cg - 2df} \\ + \overline{agg + cff} \times \overline{chh - dgh + egg - 2efh} \\ + dfh \times \overline{3agh + bgg + dff} \\ + eff \times \overline{2ahh + 3bgh - dfg + eff} \\ - efgh \times \overline{bg + 2ah} \end{matrix}\right\} = 0.$$

5 RULE.

5 RULE.

For two cubic equations.

$$ax^3 + bx^2 + cx + d = 0,$$

$$\text{and } fx^3 + gx^2 + hx + k = 0.$$

$$\text{Make } A = bf - ag,\ C = df - ak,\ D = cf - ah;$$

$$\text{and } P = cA^2 - aAC - bAD + aDD,$$

$$Q = cAC - aCC - dAD,$$

$$R = dAA - bAC + aCD.$$

$$\text{Then } PQ + RR = 0.$$

For, multiply the firſt equation by f, and the latter by a, and their difference will be found $\overline{bf - ag} \times x^2 + \overline{fc - ah} \times x + fd - ak = 0$; that is, $Ax^2 + Dx + C = 0$. And ſince $ax^3 + bx^2 + cx + d = 0$; theſe two equations come under the third rule; in which writing A, D, C for f, g, h, reſpectively; and likewiſe $cA - aC$ for A, and $bA - aD$ for D; the rule will be evident.

The Newtonian *Rule is,*

$$+\ \overline{ah - bg - 2cf} \times \overline{adhh - achk}$$

$$+\ bdfh \times \overline{ak + bh - cg - 2df}$$

$$+\ aakk \times \overline{bk - ak + 2gc + 3df}$$

$$+\ bbfk \times \overline{bk - 2dg}$$

$$+\ \overline{cdh - ddg - cck + 2bdk} \times \overline{agg + cff}$$

$$+\ \overline{3agh + bgg + dff - 3afk} \times daf$$

$$+\ bcfk \times \overline{cg + df - 3ak - bh}$$

$$-\ agk \times \overline{bbk + 3adh + cdf} = 0.$$

6 RULE.

For a cubic and a fourth power.

$$ax^4 + bx^3 + cx^2 + dx + e = 0,$$

$$\textit{and} \quad fx^3 + gx^2 + hx + k = 0.$$

$$\text{Make } A = fb - ag,\ C = fd - ak,\ D = cf - ah.$$

Then put

$P = C \times \overline{fD - gA}^2 - A \times \overline{fD - gA} \times \overline{ffe - kA}$ —

$D \times \overline{fD - gA} \times \overline{fC - hA} + A \times \overline{fC - hA}^2$,

$Q = C \times \overline{fD - gA} \times \overline{ffe - kA} - A \times \overline{ffe - kA}|^2$

$- ef \times \overline{fD - gA} \times \overline{fC - hA}$,

$R = ef \times \overline{fD - gA}^2 - D \times \overline{fD - gA} \times \overline{ffe - kA}$

$+ A \times \overline{ffe - kA} \times \overline{fC - hA}$.

Then $PQ + RR = 0$.

Or thus,

Put $E = fD - gA$, $F = ffe - kA$, $G = fC - hA$,

$P = \overline{CE - AF} \times E + \overline{AG - DE} \times G$,

$Q = \overline{CE - AF} \times F - feEG$,

$R = \overline{feE - DF} \times E + AFG$.

Then $PQ + RR = 0$, as before.

For, multiplying the firſt equation by f, and the laſt by a, the difference is $Ax^3 + Dx^2 + Cx + ef = 0$. And ſince $fx^3 + gx^2 + hx + k = 0$; it will come under Rule 5, in which write A, D, C, ef, for a, b, c, d reſpectively; and likewiſe $fD - gA$, $ffe - kA$, and $fC - hA$, for A, C, D, reſpectively; and the rule will appear.

Ex. 2.

Let $xx + 5x - 3yy = 0$,

and $3xx - 2yx + 4 = 0$,

to exterminate x.

By Rule 2, $a = 1$, $b = 5$, $c = -3yy$, $f = 3$, $g = -2y$, $h = 4$, and $A = 15 + 2y$, $B = 20 - 6y^3$, $D = -9yy - 4$.

Then $AB + DD = \overline{15 + 2y} \times \overline{20 + 6y^3} + \overline{-9yy - 4}^2 = 300 + 40y - 90y^2 - 12y^4 + 81y^4 + 72y^2 + 16 = 0$.

Ex. 3.

Suppoſe $y^3 - xyy - 3x = 0$,
and $y^2 + xy - xx + 3 = 0$,
to expunge y.

Here by Rule 3, $a = 1$, $b = -x$, $c = 0$, $d = -3x$.
and $f = 1$, $g = x$, $h = -xx + 3$.
alſo $A = -x - x = -2x$, $D = xx - 3$.

$$fD - gA = xx - 3 + 2xx = 3xx - 3,$$
$$hD - fdg = -x^4 + 6x^2 - 9 + 3x^2 = -x^4 + 9x^2 - 9$$
$$dff - hA = -3x - 2x^3 + 6x = 3x - 2x^3.$$

Then $\overline{3xx - 3} \times \overline{-x^4 + 9x^2 - 9} + \overline{3x - 2x^3}|^2 = 0$.

Or, $-3x^6 + 27x^4 - 27x^2 + 3x^4 - 27x^2 + 27 + 9x^2$
$- 12x^4 + 4x^6 = 0$.

And reduced $x^6 + 18x^4 - 45x^2 + 27 = 0$.

Ex. 4.

Let $y^4 - 3x^3y + 3 = 0$, *and* $2y^3 + xy^2 - 4x^3 = 0$, } *to expunge y.*

By Rule 6, $a = 1$, $b = 0$, $c = 0$, $d = -3x^3$, $e = 3$
$f = 2$, $g = x$, $h = 0$, $k = -4x^3$.

Then $A = -x$, $C = -2x^3$, $D = 0$. Whence $E = xx$, $F = 12 - 4x^4$, $G = -4x^3$. And

$$P = \overline{-2x^5 + 12x - 4x^5} \times xx + 4x^4 \times -4x^3$$
$$= 12x^3 - 6x^7 - 16x^7 = 12x^3 - 22x^7,$$

$$Q = \overline{-2x^5 + 12x - 4x^5} \times \overline{12 - 4x^4} + 24x^5$$
$$= \overline{12x - 6x^5} \times \overline{12 - 4x^4} + 24x^5$$
$$= 144x - 96x^5 + 24x^9,$$

$$R = 6x^4 + 4x^4 \times 12 - 4x^4 = 54x^4 - 16x^8.$$

Whence

$$PQ + RR = \overline{12x^3 - 22x^7} \times \overline{144x - 96x^5 + 24x^9}$$

$$+ \overline{54x^4 - 16x^5}|^2 = 1728x^4 - 4320x^8 + 2400x^{12} - 528x^{16} + 2916x^8 - 1728x^{12} + 256x^{16} = 0.$$

Reduced, $68x^{12} - 168x^8 + 351x^4 - 432 = 0$.

SCHOLIUM.

In the ſolution of determined problems, you will often have three or more equations, involving as many unknown quantities. Then theſe muſt be exterminated one after another, by degrees, by repeating the foregoing rules; till at laſt there remains only one unknown quantity contained in one final equation. But a perſon uſed to theſe ſorts of computations, will often find ſhorter methods than by theſe particular rules, but the finding thoſe, is only to be attained by conſtant practice.

PROBLEM LV.

To deſignate or denote any affections of literal quantities, as ſums, products, &c.

RULE.

The original quantities being written down; any affections of them, as ſums, differences, products, quotients, &c. are got by the rules of algebraic addition, ſubtraction, multiplication, diviſion, &c. before laid down.

Ex. 1.

There are two quantities, a the greater, and e the leſſer, to find the ſum, difference, product, &c. as follows.

The ſum	$a + e$
difference	$a - e$
product	$a e$
greater divided by the leſs	$\frac{a}{e}$
leſſer divided by the greater	$\frac{e}{a}$
ſum of their ſquares	$aa + ee$
difference of their ſquares	$aa - ee$
ſum of their ſum and diff.	$2a$
diff. of their ſum and diff.	$2e$
prod. of the ſum and diff.	$\overline{a+e}\times\overline{a-e}$, or $aa-ee$
ſquare of the ſum	$aa + 2ae + ee$
ſquare of the difference	$aa - 2ae + ee$
ſum of the ſquares of the ſum and difference	$2aa + 2ee$
difference of the ſquares of the ſum and diff.	$4ae$
ſquare of the product	$aaee$
cube of the greater	a^3
cube of the leſſer	e^3
cube of the ſum	$a^3 + 3a^2e + 3ae^2 + e^3$
cube of the difference	$a^3 - 3a^2e + 3ae^2 - e^3$.

Ex. 2.

There are two quantities, whoſe ſum is b, and the greater is a; what is the leſſer, the difference, &c.

Leſſer	$b - a$
difference	$2a - b$
product	$ab - aa$
greater ÷ by the leſſer	$\frac{a}{b-a}$
ſum of their ſquares	$2aa + bb - 2ba$
difference of their ſquares	$2ba - bb$
ſum of the ſum and difference	$2a$
difference of the ſum and difference	$2b - 2a$
product of the ſum and difference	$2ab - bb$
ſquare of the difference	$4aa - 4ab + bb$
difference of the ſquares of the ſum and difference	$4ab - 4aa$
ſquare of the product	$a^2b^2 - 2ba^3 + a^4$

Ex. 3.

There are two quantities the greater is a, and the greater is to the leſſer as r to s, what is the leſſer, &c.

The leſſer ($r : s :: a :$) - - - - $\frac{sa}{r}$

the ſum - - - - - - - $a + \frac{sa}{r}$

difference - - - - - - - $a - \frac{sa}{r}$

product - - - - - - - - $\frac{saa}{r}$

ſum of the ſquares - - - $aa + \frac{ssaa}{rr}$

difference of the ſquares - - $aa - \frac{ssaa}{rr}$

greater divided by the leſſer - - $\frac{r}{s}$

product of the ſum and differ. $aa - \frac{ssaa}{rr}$

ſum of the ſquares of the ſum and difference } $2aa + \frac{2ssaa}{rr}$

difference of the ſquares of the ſum and difference } - $\frac{4saa}{r}$

the ſum divided by the greater - - $1 + \frac{s}{r}$

the difference divided by the leſſer $\frac{r}{s} - 1.$

Ex. 4.

The product of two quantities is p, and the leſſer is e, what is the greater, &c.

Greater - - - - - - - $\frac{p}{e}$

ſum - - - - - - - - $\frac{p}{e} + e.$

difference - - - - - - $\frac{p}{e} - e.$

leſſer ÷ by the greater - - - $\frac{ee}{p}$

ſum of their ſquares - - - $\frac{pp}{ee} + ee$

difference of their ſquares - - - $\frac{pp}{ee} - ee$

ſum of the ſum and difference - $\frac{2p}{e}$

diff. of their ſum and diff. - - - $2e$

ſquare of the ſum - - - - $\frac{pp}{ee} + 2p + ee$

ſquare of the difference - - - $\frac{pp}{ee} - 2p + ee$

diff. ſquares of the ſum and diff. $4p$

the ſum ÷ by the difference - $\frac{p + ee}{p - ee}.$

PROBLEM LVI.

To keep a ſhort account of the ſteps in any operation.

In long and tedious operations, it is neceſſary to ſhew, how one ſtep is produced from another, or one equation derived from other foregoing ones; which to explain in words would take up a great deal of room. Therefore the method of tracing the ſeveral ſteps, will be beſt done by regiſtering them in the margin.

RULE.

Againſt every ſtep write the numbers 1, 2, 3, *&c.* in order, and ſet down, in the margin on the left hand, the ſtep or ſteps in figures, that each ſtep is produced from; with the ſigns + — ×, *&c.* according to the ſeveral operations, uſed; by which means one may ſee at one view how any equation comes, or is produced; and when an abſolute number is regiſtered, it muſt be put in a parentheſis ( ); and if any quantity is added, ſubtracted, *&c.* it muſt be put down.

Example.

Let {	1	$a + e = b.$
	2	$a - e = c.$
$1 + 2$	3	$2a = b + c$
$1 - 2$	4	$e = b - c$
1×2	5	$aa - ee = be$
$1 \div 2$	6	$\frac{a + e}{a - e} = \frac{b}{c}$
1 ɯ 2*p*	7	$\sqrt{a + e} = \sqrt{b}$
4 ⊚ 2*p*	8	$4ee = bb - 2bc + cc$
$3 + 7$	9	$2a + \sqrt{a + e} = b + c + \sqrt{b}$
4×5	10	$2aae - 2e^3 = bbc - bcc$
$3 + (4)$	11	$2a + 4 = b + c + 4$
$4 \div (4)$	12	$\frac{e}{2} = \frac{b - c}{4}.$
$9 - \sqrt{a + e}$	13	$2a = b + c + \sqrt{b} - \sqrt{a + e}$
$3 = 13$	14	$b + c = b + c + \sqrt{b} - \sqrt{a + e}.$

&c.

EXPLA-

EXPLANATION.

$1 + 2$ ſignifies that the third ſtep is found by adding the firſt and ſecond ſteps together. $1 - 2$ ſignifies, the fourth ſtep is got by ſubtracting the ſecond from the firſt. Likewiſe, the fifth ſtep (1×2) is had by multiplying the firſt and ſecond: the ſixth ſtep, by dividing the firſt by the ſecond: the ſeventh, by extracting the ſquare root of the firſt: the eighth $(4 \circledS 2p)$ is had by ſquaring the fourth: the ninth $(3 + 7)$, by adding the third and ſeventh ſteps: the tenth (4×5), by multiplying the fourth and fifth ſteps: the eleventh $(3 + (4))$, is had by adding the number 4 to the third ſtep: the twelfth $(4 \div (4))$, ſhews that it is gained by dividing the fourth ſtep by the number 4: and the thirteenth $(9 - \sqrt{a + e})$, is had by ſubtracting $\sqrt{a + e}$ from the ninth: the fourteenth $(3 = 13)$ is got by making the third and thirteenth equations equal; and ſo for others.

◆

SECT VI.

Infinite Series.

AN infinite ſeries is formed, either by actually dividing any fractional quantity having a compound denominator, or by extracting the root of a ſurd; and ſuch ſeries being continued will run on *ad infinitum*, in the manner of a decimal fraction. And in many caſes the law of the progreſſion of the terms will be evident, by obtaining a few of the foremoſt; and conſequently may be continued without actually performing the whole operation.

PROBLEM LVII.

To find the value of a fraction or ſurd, to be deſignated by an infinite ſeries.

1 RULE.

Proceed in the ſame manner as is taught in Prob. iv. Rule 2. for diviſion; or in Prob. vi. Rule 2 and 3, continuing on the operation at pleaſure.

Ex 1.

Let $\frac{ax}{a-x}$ *be given.*

$$\begin{array}{l} a-x)\ ax \qquad \left(x+\frac{xx}{a}+\frac{x^3}{aa}+\frac{x^4}{a^3}+\&c.\right. \\ \qquad ax-xx \\ \hline \qquad\quad +xx \\ \qquad\quad +xx-\frac{x^3}{a} \\ \hline \qquad\qquad\quad +\frac{x^3}{a} \\ \qquad\qquad\quad +\frac{x^3}{a}-\frac{x^4}{aa} \\ \hline \qquad\qquad\qquad\quad +\frac{x^4}{aa}\ \&c. \end{array}$$

Therefore

$$\frac{ax}{a-x}=x+\frac{xx}{a}+\frac{x^3}{aa}+\frac{x^4}{a^3}+\frac{x^5}{a^4}+\frac{x^6}{a^5}\ \&c.$$

ad infinitum.

Ex. 2.

Let the fraction $\frac{aa}{b+x}$ *be propoſed.*

$$b+x)\,aa+0\,\left(\frac{aa}{b}-\frac{aax}{bb}+\frac{aaxx}{b^3}-\frac{aax^3}{b^4}+\frac{aax^4}{b^5}\right.\ \&c$$

Anſwer.

$$\underline{aa+\frac{aax}{b}}$$

$$-\frac{aax}{b}+0$$

$$\underline{-\frac{aax}{b}-\frac{aaxx}{bb}}$$

$$+\frac{aaxx}{bb}$$

$$\underline{+\frac{aaxx}{bb}+\frac{a^2x^3}{b^3}}$$

$$-\frac{a^2x^3}{b^3}\ \&c.$$

Or thus,

$$x+b)\,aa+0\,\left(\frac{aa}{x}-\frac{aab}{x^2}+\frac{a^2b^2}{x^3}-\frac{a^2b^3}{x^4}\right.\ \&c.$$

$$\underline{aa+\frac{baa}{x}}$$

$$-\frac{baa}{x}$$

$$\underline{-\frac{aab}{x}-\frac{aabb}{xx}}$$

$$+\frac{aab^2}{xx}\ \&c.$$

Ex.

Ex. 3.

Suppose $\frac{1}{1+xx}$.

$$\begin{array}{l}
1+xx\,)\;1+0\;(\,1-xx+x^4-x^6+x^8-\&c.\\
\qquad\quad 1+xx\\
\hline
\qquad\quad -xx\\
\qquad\quad -xx-x^4\\
\hline
\qquad\qquad\quad +x^4\\
\qquad\qquad\quad +x^4+x^6\\
\hline
\qquad\qquad\qquad\quad -x^6\\
\qquad\qquad\qquad\quad -x^6-x^8\\
\hline
\qquad\qquad\qquad\qquad\quad +x^8\ \&c.
\end{array}$$

Ex. 4.

Let the fraction be $\frac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}$.

$$\begin{array}{l}
1+x^{\frac{1}{2}}-3x\,)\;2x^{\frac{1}{2}}-x^{\frac{3}{2}}+0\;(\,2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^2\\
\qquad\qquad\qquad 2x^{\frac{1}{2}}+2x-6x^{\frac{3}{2}}\qquad\qquad\qquad +34x^{\frac{5}{2}}\ \&c.\\
\hline
\qquad\qquad\qquad\quad -2x+5x^{\frac{3}{2}}\\
\qquad\qquad\qquad\quad -2x-2x^{\frac{3}{2}}+6x^2\\
\hline
\qquad\qquad\qquad\qquad\quad +7x^{\frac{3}{2}}-6x^2\\
\qquad\qquad\qquad\qquad\quad +7x^{\frac{3}{2}}+7x^2-21x\\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad -13x^2+21x^{\frac{5}{2}}\\
\qquad\qquad\qquad\qquad\qquad\qquad -13x^2-13x^{\frac{5}{2}}\\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad +34x^{\frac{5}{2}}\ \&c.
\end{array}$$

Ex. 5.

Extract the ſquare root of $aa + xx$.

$$aa + xx \left(a + \frac{xx}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \&c. = \sqrt{aa + xx} \right.$$

$$\underline{aa}$$

$$\left. 2a + \frac{xx}{2a} \right) 0 + xx$$

$$\underline{+ xx + \frac{x^4}{4aa}}$$

$$\left. 2a + \frac{xx}{a} - \frac{x^4}{8a^3} \right) - \frac{x^4}{4aa}$$

$$\underline{- \frac{x^4}{4aa} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}}$$

$$\left. 2a + \frac{xx}{a} - \frac{x^4}{8a^3} \,\&c. \right) + \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$$

$$\underline{+ \frac{x^6}{8a^4} + \frac{x^8}{16a^6} \quad \&c.}$$

$$- \frac{5x^8}{64a^6} \quad \&c.$$

Here ſuch terms are neglected whoſe dimenſions exceed thoſe of the laſt term $\frac{5x^8}{128a^7}$, to which the root is to be continued. By the ſame way it may be extracted in this form $\sqrt{xx + aa} = x + \frac{aa}{2x} - \frac{a^4}{8x^3} + \frac{a^6}{16x^5} - \&c.$

Ex. 6.

Extract the cube root of $1 - x^3$.

From $1 - x^3 \left(1 - \frac{x^3}{3} - \frac{x^6}{9} - \frac{5x^9}{81} \right.$ &c.
take 1

$3) - x^3$

From $1 - x^3$
take $1 - x^3 + \frac{x^6}{3} - \frac{x^9}{27} = \overline{1 - \frac{x^3}{3}}\Big|^3$.

$3) - \frac{x^6}{3} + \frac{x^9}{27}$

From $1 - x^3$
take $1 - x^3 \; * + \frac{5x^9}{27}$ &c. $= \overline{1 - \frac{x^3}{3} - \frac{x^6}{9}}\Big|^3$.

$3) - \frac{5x^9}{27}$ &c.

2 RULE.

Aſſume a ſeries with unknown coefficients, to repreſent it. Which ſeries being multiplied, or involved, *&c* according as the queſtion requires; the quantities of the ſame dimenſion muſt be put equal to each other; from which equations, the coefficients will be determined.

Ex. 7.

Let $\frac{1}{a - x}$ *be given.*

Suppoſe $\frac{1}{a - x} = A + Bx + Cx^2 + Dx^3 + Ex^4$ &c. the ſeries required. Multiply by $a - x$. Then

$$\begin{aligned} 1 = aA &+ aBx + aCx^2 + aDx^3 + aEx^4 \text{ \&c.} \\ &- Ax - Bx^2 - Cx^3 - Dx^4 \text{ \&c.} \end{aligned}$$

Whence equating the coefficients of the ſame powers of x, we have $aA = 1$, $aB - A = 0$, $aC - B = 0$, $aD - C = 0$, $aE - D = 0$, &c. Therefore $A = \frac{1}{a}$, $B = \frac{A}{a} = \frac{1}{aa}$, $C = \frac{B}{a} = \frac{1}{a^3}$, $D = \frac{C}{a} = \frac{1}{a^4}$, $E = \frac{D}{a} = \frac{1}{a^5}$, &c. by reduction. Therefore the ſeries is $\frac{1}{a} + \frac{x}{aa} + \frac{x^2}{a^3}$ &c. or $\frac{1}{a - x} = \frac{1}{a} + \frac{x}{aa} + \frac{x^2}{a^3} + \frac{x^3}{a^4} + \frac{x^4}{a^5} +$ &c.

Ex. 8.

Let $\frac{cc}{cc + 2cy - yy}$ *be given.*

Suppoſe it $= A + By + Cy^2 + Dy^3$ &c. Multiply by $cc + 2cy - yy$. Then

$$\begin{aligned} cc = ccA + ccBy &+ ccCy^2 + ccDy^3 \text{ \&c.} \\ + 2cAy &+ 2cBy^2 + 2cCy^3 \\ &- Ay^2 - By^3 \end{aligned}$$

And equating the homologous terms, $cc = ccA$, $ccB + 2cA = 0$, $ccC + 2cB - A = 0$, $ccD + 2cC - B = 0$, &c. and by reduction, $A = 1$, $B = -\frac{2A}{c} = -\frac{2}{c}$, $C = \frac{A - 2cB}{cc} = \frac{1 + 4}{cc} = \frac{5}{cc}$, $D = \frac{B - 2cC}{cc} = \frac{-2 - 10}{c^3} = -\frac{12}{c^3}$, &c. Whence $\frac{cc}{cc + 2cy - yy} = 1 - \frac{2y}{c} + \frac{5y^2}{cc} - \frac{12y^3}{c^3}$ &c.

Ex. 9.

What is $\sqrt{aa-xx}$.

Let $\sqrt{aa-xx} = A + Bx^2 + Cx^4 + Dx^6$ &c. which being ſquared,

$$\begin{array}{l} aa - xx = A^2 + 2ABx^2 + B^2x^4 + 2ADx^6 \text{ \&c.} \\ \qquad\qquad\qquad\qquad\quad + 2ACx^4 + 2BCx^6 \end{array}$$

Here $A^2 = aa$, $2AB = -1$, $BB + 2AC = 0$. $2AD + 2BC = 0$, *&c.* Whence $A = a$, $B = -\frac{1}{2A} = -\frac{1}{2a}$, $C = -\frac{BB}{2A} = -\frac{1}{8a^3}$, $D = -\frac{BC}{A} = -\frac{1}{16a^5}$, *&c.* Therefore $\sqrt{aa-xx}$ $= a - \frac{xx}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$ &c.

PROBLEM LVIII.

To reduce any binomial ſurd to an infinite ſeries, or to extract any root of a binomial.

RULE.

This is done by ſubſtituting the particular letters or quantities, inſtead of theſe in the following general form, duly obſerving the ſigns.

$$\overline{P+PQ}|^{\frac{m}{n}} = \overset{A}{P^{\frac{m}{n}}} + \overset{B}{\frac{m}{n}AQ} + \overset{C}{\frac{m-n}{2n}BQ}$$

$$+ \overset{D}{\frac{m-2n}{3n}CQ} + \overset{E}{\frac{m-3n}{4n}DQ} + \text{\&c.}$$

Where P is the firſt term, Q the ſecond term divided by the firſt, $\frac{m}{n}$ the index of the power or root, A, B, C, D, *&c.* the foregoing terms with their ſigns.

Ex.

Ex. 1.

Extract the ſquare root of rr — xx.

Here $P = rr$, $Q = \frac{-xx}{rr}$, $\frac{m}{n} = \frac{1}{2}$. Therefore $\overline{rr - xx}|^{\frac{1}{2}} = r + \frac{1}{2}A \times \frac{-xx}{rr} - \frac{1}{4}B \times \frac{-xx}{rr} - \frac{3}{6}C \times \frac{-xx}{rr} - \frac{5}{8}D \times \frac{-xx}{rr} - \&c. = r - \frac{x}{2rr}A + \frac{xx}{4rr}B + \frac{3xx}{6rr}C + \frac{5xx}{8rr}D + \&c.$ that is, reſtoring the values of A, B, C, &c. $\sqrt{rr - xx} = r - \frac{xx}{2r} - \frac{x^4}{8r^3} - \frac{x^6}{16r^5} - \frac{5x^8}{128r^7}$ &c.

Ex. 2.

What is the value of $\frac{rr}{r + x}$?

Here $\frac{rr}{r+x} = rr \times \overline{r + x}|^{-1}$, and $P = r$, $Q = \frac{x}{r}$, $\frac{m}{n} = -1$, or $m = -1$, $n = 1$. Therefore $\overline{r + x}|^{-1} = r^{-1} - 1A \times \frac{x}{r} - 1B \times \frac{x}{r} - 1C \times \frac{x}{r} - 1D \times \frac{x}{r}$ &c. $= \frac{1}{r} - \frac{x}{r}A - \frac{x}{r}B - \frac{x}{r}C$ &c. And $rr \times \overline{r + x}|^{-1} = rr \times : \frac{1}{r} - \frac{x}{rr} + \frac{xx}{r^3} - \frac{x^3}{r^4}$ &c : that is, $\frac{rr}{r + x} = r - x + \frac{xx}{r} - \frac{x^3}{r^2} + \frac{x^4}{r^3}$ &c.

Ex.

Ex. 3.

To find the value of $\frac{1}{\sqrt{2rx - xx}}$.

$\frac{1}{\sqrt{2rx - xx}} = \overline{2rx - xx}|^{-\frac{1}{2}}$, and $P = 2rx$, $Q = -\frac{x}{2r}$, $m = -1$, $n = 2$. Then

$$\overline{2rx - xx}|^{-\frac{1}{2}} = \overline{2rx}|^{-\frac{1}{2}} - \frac{1}{2}A \times \frac{-x}{2r} - \frac{3}{4}B \times \frac{-x}{2r} - \frac{5}{6}C \times \frac{-x}{2r} - \frac{7}{8}D \times \frac{-x}{2r} \text{ \&c.} = \frac{1}{\sqrt{2rx}} + \frac{x}{4r}A + \frac{3x}{8r}B + \frac{5x}{12r}C + \frac{7x}{16r}D + \text{\&c.}$$

$$= \frac{1}{\sqrt{2rx}} + \frac{x}{4r\sqrt{2rx}} + \frac{3xx}{32rr\sqrt{2rx}} \text{ \&c.} = \frac{1}{\sqrt{2rx}} \times : 1 + \frac{x}{4r} + \frac{3x^2}{32r^2} + \frac{3 \cdot 5x^3}{4.8.12r^3} + \frac{3 \cdot 5 \cdot 7x^4}{4.8.12.16r^4} + \text{\&c.}$$

Ex. 4.

What is the cube root of $1 - x^3$.

Here $P = 1$, $Q = -x^3$, $m = 1$, $n = 3$. Whence

$$\overline{1 - x^3}|^{\frac{1}{3}} = 1 + \frac{1}{3}A \times -x^3 - \frac{2}{6}B \times -x^3 - \frac{5}{9}C \times -x^3 - \frac{8}{12}D \times -x^3 - \frac{11}{15}E \times -x^3 \text{ \&c.} = 1 - \frac{x^3}{3}A + \frac{x^3}{3}B + \frac{5x^3}{9}C + \frac{2x^3}{3}D + \frac{11x^3}{15}E$$

&c. that is, $\sqrt[3]{1 - x^3} = 1 - \frac{x^3}{3} - \frac{x^6}{9} - \frac{5x^9}{81} - \frac{10x^{12}}{243} - \frac{22x^{15}}{729}$ &c.

Ex. 5.

What is $\sqrt[3]{\frac{aa}{\overline{aa+xx}|^{2}}}$ *in an infinite ſeries.*

This reduced is $a^{\frac{2}{3}} \times \overline{aa+xx}|^{-\frac{2}{3}}$. Here P = aa, Q = $\frac{xx}{aa}$, $m = -2$, $n = 3$. And $\overline{aa+xx}|^{-\frac{2}{3}}$ $= \overline{aa}|^{-\frac{2}{3}} - \frac{2}{3} A \frac{xx}{aa} - \frac{5}{6} B \frac{xx}{aa} - \frac{8}{9} C \frac{xx}{aa} - \frac{11}{12} D \frac{xx}{aa}$ &c. $= \frac{1}{a^{1\frac{1}{3}}} - \frac{2xx}{3a^{3\frac{1}{3}}} + \frac{5x^4}{9a^{5\frac{1}{3}}} - \frac{40x^6}{81a^{7\frac{1}{3}}} + \frac{110x^8}{243a^{9\frac{1}{3}}}$ &c. $= \frac{1}{a^{\frac{1}{3}}} \times : \frac{1}{a} - \frac{2x^2}{3a^3} - \frac{5x^4}{9a^5} - \frac{40x^6}{81a^7} + \frac{110x^8}{243a^9}$ — &c: therefore $\frac{a^{\frac{2}{3}}}{\overline{aa+xx}|^{\frac{2}{3}}}$ $= \frac{1}{\sqrt[3]{aa}} \times : 1 - \frac{2x^2}{3a^2} + \frac{5x^4}{9a^4} - \frac{40x^6}{81a^6}$ + &c.

Ex. 6.

What is the value of $\sqrt[5]{aa-xx}$.

$\sqrt[5]{aa-xx} = \overline{aa-xx}|^{\frac{1}{5}}$. Here P = aa, Q = $\frac{-xx}{aa}$, $m = 1$, $n = 5$. Therefore $\overline{aa-xx}|^{\frac{1}{5}}$ $= \overline{aa}|^{\frac{1}{5}} + \frac{1}{5} A \times \frac{-xx}{aa} - \frac{4}{10} B \times \frac{-xx}{aa} - \frac{9}{15} C \times \frac{-xx}{aa} - \frac{14}{20} D \times \frac{-xx}{aa}$ &c. $= a^{\frac{2}{5}} - \frac{xx}{5aa} A + \frac{2xx}{5aa} B + \frac{3xx}{5aa} C + \frac{7xx}{10aa} D$ &c. $= a^{\frac{2}{5}} \times : 1 - \frac{xx}{5aa} - \frac{2x^4}{25a^4} - \frac{6x^6}{125a^6} - \frac{21x^8}{625a^8}$ &c.

Ex. 7.

To reduce $\overline{a+x}\times\sqrt[4]{\overline{a-x}}$ *to a ſeries.*

$\sqrt[4]{\overline{a-x}}=\overline{a-x}|^{\frac{1}{4}}$. Where $P=a$, $Q=\frac{-x}{a}$, $m=1$, $n=4$. Then $\overline{a-x}|^{\frac{1}{4}}=a^{\frac{1}{4}}+\frac{1}{4}A\times\frac{-x}{a}-\frac{3}{8}B\times\frac{-x}{a}-\frac{7}{12}C\times\frac{-x}{a}$ &c. $=a^{\frac{1}{4}}-\frac{x}{4a}A+\frac{3x}{8a}B+\frac{7x}{12a}C$ &c. $=$

$$a^{\frac{1}{4}}-\frac{x}{4a^{\frac{3}{4}}}-\frac{3x^2}{32a^{\frac{1}{4}}}-\frac{7x^3}{128a^{\frac{11}{4}}}\text{ \&c.}$$

Multiply by $a+x$

$$\text{Then}\quad a^{\frac{5}{4}}-\frac{a^{\frac{1}{4}}x}{4}-\frac{3x^2}{32a^{\frac{3}{4}}}-\frac{7x^3}{128a^{\frac{7}{4}}}\text{ \&c.}$$

$$+a^{\frac{1}{4}}x-\frac{x^2}{4a^{\frac{3}{4}}}-\frac{3x^3}{32a^{\frac{7}{4}}}\text{ \&c.}$$

$$\overline{a+x}\times\sqrt[4]{\overline{a-x}}=a^{\frac{5}{4}}+\frac{3a^{\frac{1}{4}}x}{4}-\frac{11x^2}{32a^{\frac{3}{4}}}-\frac{19x^3}{128a^{\frac{7}{4}}}\text{ \&c.}$$

$$=\sqrt[4]{a}\times:a+\frac{3x}{4}-\frac{11x^2}{32a}-\frac{19x^3}{128aa}-\text{\&c.}$$

Ex. 8.

To deſignate $\sqrt{\frac{aa+xx}{aa-xx}}$ *by a ſeries.*

$\sqrt{aa+xx} = \overline{aa+xx}|^{\frac{1}{2}}$. Where P $= aa$, Q $= \frac{xx}{aa}$, $m = 1$, $n = 2$, and $\overline{aa+xx}|^{\frac{1}{2}} = a + \frac{1}{2}A\,\frac{xx}{aa} - \frac{1}{4}B\,\frac{xx}{aa} - \frac{3}{6}C\,\frac{xx}{aa}$ &c. $= a + \frac{xx}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}$ &c.

Again, $\frac{1}{\sqrt{aa-xx}} = \overline{aa-xx}|^{-\frac{1}{2}}$. Here P $= aa$, Q $= \frac{-xx}{aa}$, $m = -1$, $n = 2$. And $\overline{aa-xx}|^{-\frac{1}{2}} = \frac{1}{a} - \frac{1}{2}A \times \frac{-xx}{aa} - \frac{3}{4}B \times \frac{-xx}{aa} - \frac{5}{6}C \times \frac{-xx}{aa}$ &c. $= \frac{1}{a} + \frac{xx}{2a^3} + \frac{3x^4}{8a^5} + \frac{5x^6}{16a^7}$ &c. Whence $\sqrt{\frac{aa+xx}{aa-xx}}$ or $\frac{\sqrt{aa+xx}}{\sqrt{aa-xx}} = a + \frac{xx}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}$ &c. $\times \frac{1}{a} + \frac{xx}{2a^3} + \frac{3x^4}{8a^5} + \frac{5x^6}{16a^7}$ &c. $= 1 + \frac{xx}{aa} - \frac{x^4}{2a^4} + \frac{x^6}{2a^6}$ &c. by multiplication.

Ex.

Ex. 9.

What is the value of $\frac{ax}{aa - ax + xx}$.

This may be treated as a binomial. Put $y = ax - xx$. Then $aa - ax + xx = aa - y$. And $\frac{ax}{aa - ax + xx} = ax \times \overline{aa - y}|^{-1}$. Here $P = aa$, $Q = \frac{-y}{aa}$, $m = -1$, $n = 1$.

And $\overline{aa - y}|^{-1} + \frac{1}{aa} - 1A \times \frac{-y}{aa} - 1B \times \frac{-y}{aa} - 1C \times \frac{-y}{aa} - 1D \times \frac{-y}{aa}$ &c. $= \frac{1}{aa} + \frac{y}{aa}A + \frac{y}{aa}B + \frac{y}{aa}C + \frac{y}{aa}D$ *&c.* $= \frac{1}{aa} + \frac{y}{a^4} + \frac{y^2}{a^6} + \frac{y^3}{a^8} + \frac{y^4}{a^{10}}$ &c. = (by reſtitution) $\frac{1}{aa} + \frac{ax - xx}{a^4} + \frac{\overline{ax - xx}|^2}{a^6} + \frac{\overline{ax - xx}|^3}{a^8} + \frac{\overline{ax - xx}|^4}{a^{10}}$ &c. which involved and reduced into order will be

$$\begin{array}{cccccc}
\frac{1}{aa} & + \frac{x}{a^3} & - \frac{xx}{a^4} & & & \\
 & & + \frac{xx}{a^4} & - \frac{2x^3}{a^5} & + \frac{x^4}{a^6} & \\
 & & & + \frac{x^3}{a^5} & - \frac{3x^4}{a^6} & \text{\&c.} \\
 & & & & + \frac{x^4}{a^6} & \text{\&c.} \\
\hline
\frac{1}{aa} & + \frac{x}{a^3} & * & - \frac{x^3}{a^5} & - \frac{x^4}{a^6} & \text{\&c.}
\end{array}$$

and $\frac{ax}{aa - ax + xx} = \frac{x}{a} + \frac{x^2}{a^2} * - \frac{x^4}{a^4} - \frac{x^5}{a^5}$ &c.

The truth of this rule will appear by induction. For if any of theſe ſeries be involved according to the index of the root, it will produce the original quantity. Thus if $r - \frac{xx}{2r} - \frac{x^4}{8r^3}$ &c. be ſquared, it produces $rr - xx$, as in Examp. 1. If $1 - \frac{x^3}{3} - \frac{x^6}{9}$ &c. be cubed it produces $1 - x^3$, Ex. 4. If $a^{\frac{2}{5}} \times : 1 - \frac{xx}{5aa} - \frac{2x^4}{25a^4}$ &c. be involved to the 5th power, it gives $aa - xx$, Ex. 5. and the like of others.

Cor. 1. $\overline{P + PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} \times : 1 + \frac{m}{n}Q + \frac{m}{n} \times \frac{m-n}{2n}Q^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}Q^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}Q^4$ &c.

Cor. 2. $\overline{a + x}|^n = a^n + \frac{nx}{a+x}A + \frac{n+1}{2} \times \frac{x}{a+x}B + \frac{n+2}{3} \times \frac{x}{a+x}C + \frac{n+3}{4} \times \frac{x}{a+x}D +$ &c. *where n is any index; A, B, C, &c. the foregoing terms with their ſigns.*

For put $y = \frac{x}{a+x}$, then $x = \frac{ay}{1-y}$, and $a + x = \frac{a}{1-y}$. Therefore $\overline{a + x}|^n = \overline{\frac{a}{1-y}}|^n = a^n \times \overline{1-y}|^{-n}$. Here $P = 1$, $Q = -y$, $m = -n$, $n = 1$ (ſee Prob. xlix); then by this problem, $\overline{1-y}|^{-n} = 1 - \frac{n}{1}A \times -y - \frac{n+1}{2}B \times -y - \frac{n+2}{3}C \times -y - \frac{n+3}{4}D \times -y$ &c. $=$

$1 + nyA$

$1 + nyA + \frac{n+1}{2}By + \frac{n+2}{3}Cy + \frac{n+3}{4}Dy$ &c. and $a^n \times \overline{1-y}|^{-n} = a^n \times : 1 + nyA + \frac{n+1}{2}By + \frac{n+2}{3}Cy$ &c. $=$ (reſtoring the value of y) $a^n \times : 1 + \frac{nx}{a+x}A + \frac{n+1}{2} \times \frac{x}{a+x}B + \frac{n+2}{3} \times \frac{x}{a+x}C + \frac{n+3}{4} \times \frac{x}{a+x}D$ *&c.*

PROBLEM LIX.

To involve the ſeries $z \times : a + bx + cx^2 + dx^3 + ex^4$ &c. *to any power* m, *whole or fractional.*

RULE.

Subſtitute the particular letters or numbers in the given ſeries, inſtead of theſe in the following general form.

$$z^m \times \overline{a + bx + cx^2 + dx^3 + ex^4 \text{ \&c.}}|^m$$

$$= z^m \times \text{ into : } a^m$$

$$+ \frac{mbA}{a}x$$

$$+ \frac{2mcA + \overline{m-1}\,.\,bB}{2a}x^2$$

$$+ \frac{3mdA + \overline{2m-1}\,.\,cB + \overline{m-2}\,.\,bC}{3a}x^3$$

$$+ \frac{4meA + \overline{2m-1}\,.\,dB + \overline{2m-2}\,.\,cC + \overline{m-3}\,.\,bD}{4a}x^4$$

$$+ \frac{5mfA + \overline{4m-1}\,.\,eB + \overline{3m-2}\,.\,dC + \overline{2m-3}\,.\,cD + \overline{m-4}\,.\,bE}{5a}x^5$$

$$+\frac{6mgA+\overline{5m-1}\,.\,fB+\overline{4m-2}\,.\,eC+\overline{3m-3}\,.\,dD+\overline{2m-4}\,.\,cE+\overline{m-5}\,.\,bF}{6a}x^6$$

$$+\frac{7mhA+\overline{6m-1}\,.\,gB+\overline{5m-2}\,.\,fC+\overline{4m-3}\,.\,eD+\overline{3m-4}\,.\,dE+\overline{2m-5}\,.\,cF+\overline{m-6}\,.\,bG}{7a}x^7\ \&c.$$

Where A, B, C, D, *&c.* are the coefficients of the terms immediately preceding thoſe wherein they firſt appear. And the law of progreſſion is evident.

Ex. I.

What is the ſquare of $1+x+xx+x^3+x^4+$ &c.

Here $z=1$, $a=1$, $b=1$, $c=1$, $d=1$, &c. And $m=2$, then $\overline{1+x+x^2+x^3\ \&c}|^2 =$

$$1+\frac{2A}{1}x+\frac{4A+B}{2}x^2+\frac{6A+3B+0}{3}x^3+\frac{8A+5B+2C-D}{4}x^4\ \&c.$$

$$=\overset{A}{1}+\overset{B}{2x}+\overset{C}{3x^2}+\overset{D}{4x^3}+\overset{E}{5x^4}\ \&c.$$

Ex. 2.

What is the ſquare root of $1+x+xx+x^3$ &c.

Here $z=1$, $a=1$, $b=1$, $c=1$, $d=1$ &c. and $m=\frac{1}{2}$. Whence $\overline{1+x+x^2+x^3+x^4\ \&c.}|^{\frac{1}{2}}$

$$=1+\frac{1}{2}Ax+\frac{A-\frac{1}{2}B}{2}x^2+\frac{A+0-C}{3}x^3+\frac{2A+\frac{1}{2}B-C-\frac{5}{2}D}{4}x^4,\ \&c.=1+\frac{1}{2}x+\frac{3}{8}xx+\frac{5}{16}x^3+\frac{35}{128}x^4\ \&c.$$

Ex. 3.

Find the cube of $1 + x + x^2 + x^3$ &c.

Here $z = 1$, $a = 1$, $b = 1$, $c = 1$, $d = 1$, $e = 1$, &c. $m = 3$. Then $\overline{1 + x + x^2 + x^3 \text{ \&c.}}\big|^3 = 1 + \frac{3Ax}{1} + \frac{6A + 2B}{2}x^2 + \frac{9A + 5B + C}{3}x^3 + \frac{12A + 8B + 4C}{4}x^4$ &c. $= 1 + 3x + 6x^2 + 10x^3 + 15x^4 +$ &c.

Ex. 4.

What is the value of

$$\frac{1}{rr - \frac{1}{2}yy + \frac{y^4}{4r^2} - \frac{y^6}{8r^4} + \frac{y^8}{16r^6} \text{ \&c.}}$$

Here $z = 1$, $x = yy$, $a = rr$, $b = -\frac{1}{2}$, $c = \frac{1}{4rr}$, $d = -\frac{1}{8r^4}$, $e = \frac{1}{16r^6}$, &c. $m = -1$. Then

$$\overline{rr - \frac{1}{2}yy + \frac{y^4}{4r^2} - \frac{y^6}{8r^6} \text{ \&c.}}\Big|^{-1} = \frac{1}{rr} + \frac{A}{2rr}x + \frac{-\frac{1}{2rr}A + B}{2rr}xx + \frac{\frac{3}{8r^4}A - \frac{3}{4rr}B + \frac{3}{2}C}{3rr}x^3 \text{ \&c.}$$

$$= \frac{1}{rr} + \frac{1}{2r^4}x + 0x^2 + 0x^3 \text{ \&c.} = \frac{1}{rr} + \frac{1}{2r^4}x = \frac{1}{rr} + \frac{1}{2r^4}yy.$$

Ex. 5.

To ſquare the ſeries $y - y^3 + y^5 - y^7 + y^9$ &c.

This is equal to $y \times : 1 - y^2 + y^4 - y^6 + y^3$ &c. Here $z = y$, $x = yy$, $a = 1$, $b = -1$, $c = 1$, $d = -1$, $e = 1$,

$e = 1$, &c. and $m = 2$. Then $\overline{1 - y^2 + y^4 \text{ &c.}}|^2 =$ $1 - 2Ax + \frac{4A - B}{2} x^2 + \frac{-6A + 3B}{3} x^3$ &c. $=$ $- 2x + 3x^2 - 4x^3$ &c. and $\overline{y - y^3 - y^5 \text{ &c.}}|^2 =$ $y^2 \times \overline{1 - 2x + 3x^2 \text{ &c.}} = yy - 2y^4 + 3y^6 - 4y^8 +$ &c.

Ex. 6.

To ſquare the ſeries.

$$\sqrt{2r} \times : v^{\frac{1}{2}} + \frac{v^{\frac{2}{3}}}{2.2.3r} + \frac{3v^{\frac{5}{2}}}{4.2.4.5r^2} + \frac{3.5.v^{\frac{7}{2}}}{8.2.4.6.7r^3} \text{ &c.}$$

The ſeries is $\overline{2rv}|^{\frac{1}{2}} \times : 1 + \frac{v}{12r} + \frac{3v^2}{160r^2} + \frac{5v^3}{896r^3}$ &c. Here $z = \sqrt{2rv}$, $a = 1$, $b = \frac{1}{12r}$, $c = \frac{3}{160rr}$, $d = \frac{5}{896r^3}$, &c. $m = 2$. Then

$$\overline{\overline{2rv}|^{\frac{1}{2}} \times : 1 + \frac{v}{12r} + \frac{3v^2}{160r^2} \text{ &c.}}\Big|^2 = 2rv \times : 1 + + \frac{1}{6r} A x + \frac{\frac{3}{40rr} A + \frac{1}{12r} B}{2} x^2 + \frac{\frac{30}{896r^3} A + \frac{9}{160rr} B}{3} x^3 \text{ &c} = 2rv \times : 1 + \frac{x}{6r} + \frac{2x^2}{45rr} + \frac{57x^3}{4480r^3} \text{ &c.}$$

Ex. 7.

Find the m power of

$$ax^r + bx^{r+n} + cx^{r+2n} + dx^{r+3n} \text{ &c.}$$

This reduced is $x^r \times : a + bx^n + cx^{2n} + dx^{3n}$ &c. Here $z = x^r$, $x = x^n$, $m = 2$, &c.

Then $\overline{x^r \times : a + bx^n + cx^{2n} \ \&c.}|^m = x^{rm} \times : a^m$.

$$+ \frac{mb}{a}Ax^n + \frac{2mcA + \overline{m-1}\,.\,bB}{2a}x^{2n} +$$

$$\frac{3mdA + \overline{2m-1}\,.\,cB + \overline{m-2}\,.\,bC}{3a}x^{3n} +$$

$$\frac{4meA + \overline{3m-1}\,.\,dB + \overline{2m-2}\,.\,cC + \overline{m-3}\,.\,bD}{4a}x4^{n}$$

&c.

2 RULE.

Subſtitute each letter in the given ſeries, inſtead of the correſpondent one, in the following general form.

$$z^m \times \overline{a + bx + cx^2 + dx^3 + ex^4 \ \&c.}|^m = z^m \times \text{into}$$

$$a^m + mba^{m-1}x + \left.\begin{matrix} m\,.\,\frac{m-1}{2}a^{m-2}bb \\ + ma^{m-1}c \end{matrix}\right\} x^2$$

$$\left.\begin{matrix} + m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}a^{m-3}b^3 \\ + m\,.\,\frac{m-1}{2}\,.\,2a^{m-2}bc \\ + ma^{m-1}d \end{matrix}\right\} x^3$$

$$\left.\begin{matrix} + m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,.\,\frac{m-3}{4}a^{m-4}b^4 \\ + m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,.\,3a^{m-3}bbc \\ + m\,.\,\frac{m-1}{2}\,.\,a^{m-2} \times \left\{\begin{matrix} 2bd \\ + cc \end{matrix}\right. \\ + ma^{m-1}e \end{matrix}\right\} x4$$

$$\left.\begin{array}{l} + m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}\cdot\frac{m-4}{5}a^{m-5}\,b^5 \\ \quad + m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}\cdot 4a^{m-4}\,b^3c \\ \qquad + m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot 3a^{m-3}\left\{\begin{array}{l} bcc \\ +bbd \end{array}\right. \\ \qquad\quad + m\cdot\frac{m-1}{2}\cdot 2a^{m-2}\left\{\begin{array}{l} cd \\ be \end{array}\right. \\ \qquad\qquad + ma^{m-1}\,f \end{array}\right\} x^5$$

$$\left.\begin{array}{l} + m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}\cdot\frac{m-4}{5}\cdot\frac{m-5}{6}a^{m-6}\,b^6 \\ \quad + m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}\cdot\frac{m-4}{5}\cdot 5a^{m-5}\,b^4c \\ \qquad + m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}\cdot a^{m-4}\left\{\begin{array}{l} 6bbcc \\ 4b^3d \end{array}\right. \\ \qquad\quad + m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot a^{m-3}\left\{\begin{array}{l} 3bbe \\ 6bcd \\ c^3 \end{array}\right. \\ \qquad\qquad + m\cdot\frac{m-1}{2}\,a^{m-2}\left\{\begin{array}{l} 2bf \\ 2ce \\ dd \end{array}\right. \\ \qquad\qquad\quad + ma^{m-1}\,g \end{array}\right\} x^6$$

&c.

For let $y = bx + cx^2 + dx^3$ &c. $p = \frac{m-1}{2}m$, $q = \frac{m-2}{3}p$, $r = \frac{m-3}{4}q$, $s = \frac{m-4}{5}r$, &c. Then $\overline{a + bx + cx^2 + dx^3}|^m$ &c. $= \overline{a+y}|^m = a^m + ma^{m-1}y + pa^{m-2}y^2 + qa^{m-3}y^3 + ra^{m-4}y^4$ &c. But

$$y = bx + cx^2 + dx^3 + ex^4 \text{ \&c.}$$

$$yy = bbxx + 2bcx^3 \begin{array}{l} +2bd \\ +cc \end{array} x^4 \begin{array}{l} +2be \\ +2cd \end{array} x^5 \text{ \&c.}$$

$y^3 = b^3$

$$\begin{array}{llllll}
y^3 & = b^3x^3 & + 3bbcx^4 & + 3bbd\,x^5 & + \&c. \\
 & & & + 3bcc & \\
y^4 & = b^4x^4 & + 4b^3cx^5 & + \&c. & \\
y^5 & = b^5x^5 & & & \\
\&c. & & & &
\end{array}$$

Then the power $a^m + ma^{m-1}y + pa^{m-2}yy$ $qa^{m-3}y^3$ &c. becomes

$$\begin{array}{lllllll}
a^m & & & & & & \\
+ ma^{m-1} \times : & bx & + cx^2 & + dx^3 & + ex^4 & + fx^5 & \&c. \\
+ pa^{m-2} \times : & & bbx^2 & + 2bcx^3 & + 2bdx^4 & + 2bex^5 & \\
 & & & & + cc & + 2cd & \\
+ qa^{m-3} \times : & & & b^3x^3 & + 3b^2cx^4 & + 3bbdx^5 & \\
 & & & & & + 3bcc & \\
+ ra^{m-4} \times : & & & & b^4x^4 & + 4b^3cx^5 & \\
+ sa^{m-5} \times : & & & & & b^5x^5 & \\
\&c. & & & & & &
\end{array}$$

Theſe being actually multiplied, and the coefficients of each power of x collected; will give the ſeveral terms as in the form above.

And the firſt Rule is in effect the ſame as this: For let $\overline{a + bx + cxx + dx^3 \ \&c.}|^m = A + Bx^2 + Cx^3 + Dx^4$ &c. Then by Rule 1, $A = a^m$, as in Rule 2d. Alſo $B = \frac{mbA}{a} = mba^{m-1}$, as in Rule 2d. Likewiſe $C = \frac{mcA}{a} + \frac{\overline{m-1}}{2a} . bB = mca^{m-1} + \frac{\overline{m-1}}{2} . mbba^{m-2}$, as in Rule 2d.

Again $D = \frac{mdA}{a} + \frac{\overline{2m-1}\,.\,cB}{3a} + \frac{\overline{m-2}\,.\,bC}{3a}$

$$= mda^{m-1} + \frac{2m-1}{3} \times mcba^{m-2} + \overline{\frac{m-2}{3}}\,.\,b \times$$

$$\overline{mca^{m-2} + \frac{m-1}{2}\,.\,mbba^{m-3}} = mda^{m-1} +$$

$$\overline{\frac{2m-1}{3} + \frac{m-2}{3}} \times mbca^{m-2} + m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,b^3$$

$$a^{m-3} = mda^{m-1} + m\,.\,\frac{m-1}{2}\,.\,2bca^{m-2} +$$

$+ m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,b^3a^{m-3}$, as in Rule 2d, and ſo for the reſt. In uſing this laſt rule, it will be the eaſieſt way to divide all by the firſt term, that a may be 1.

Ex. 8.

What is the fourth power of $1 + x + x^2 + x^3$ &c.

Here $z = 1$, $a = 1$, $b = 1$, $c = 1$, $d = 1$, &c. $m = 4$. Then $\overline{1 + x + x^2 + x^3 \text{ \&c.}}|^4$

$$\begin{array}{llllll}
= 1 + 4bx & + 6b^2x^2 & + 4b^3x^3 & + b^4x^4 & \text{\&c.} \\
 & + 4c & + 12bc & + 12bbc & \\
 & & + 4d & + 6cc & \\
 & & & + 12bd & \\
 & & & + 4e & \\
= 1 + 4x & + 10x^2 & + 20x^3 & + 35x^4 & \text{\&c.}
\end{array}$$

Ex. 9.

What is the ſquare of $\frac{1}{x} + \frac{1}{xx} + \frac{1}{x^3} + \frac{1}{x^4}$ &c.

In this Example, $z = \frac{1}{x}$, $x = \frac{1}{x}$, $a = 1$, $b = 1$, $c = 1$, $d = 1$, &c. $m = 2$.

Then $\overline{\frac{1}{x}+\frac{1}{xx}+\frac{1}{x^3}\ \&c.}\Big|^2=\frac{1}{xx}\times:$

$$1+2b\times\frac{1}{x}+\begin{matrix}bb\\+2c\end{matrix}\times\frac{1}{xx}+\begin{matrix}2bc\\+2d\end{matrix}\times\frac{1}{x^3}+\begin{matrix}2bd\\+cc\\+2e\end{matrix}\times\frac{1}{x^4}\ \&c.$$

$$=\frac{1}{xx}\times:1+\frac{2}{x}+\frac{3}{xx}+\frac{4}{x^3}+\frac{5}{x^4}\ \&c.=\frac{1}{xx}$$
$$+\frac{2}{x^3}+\frac{3}{x^4}+\frac{4}{x^5}+\frac{5}{x^6}\ \&c.$$

Ex. 10.

To ſquare the ſeries $y-y^3+y^5-y^7+\&c.$

Here $z=y$, $a=1$, $b=0$, $c=-1$, $d=0$, $e=1$, $f=0$, $g=-1$ &c. and $m=2$. Whence

$$\overline{y-y^3+y^5\ \&c.}\big|^2=y^2\times:$$
$$1+0x-2ax^2-0x^3+\begin{matrix}ccx^4\\+2e\end{matrix}$$
$$+0x^5-\begin{matrix}2cex^6\\-2g\end{matrix}\ \&c.=y^2\times:1-2y^2+3y^4-4y^6\ \&c.$$
$$=y^3-2y^4+3y^6-4y^8\ \&c.$$

Or thus,

$z=y$, $x=yy$, $a=1$, $b=-1$, $c=1$, $d=-1$, &c. and $m=2$. Then $\overline{y-y^3+y^5\ \&c.}^2=yy\times:$

$$1+2bx+\begin{matrix}bbx^2\\+2c\end{matrix}+\begin{matrix}2bcx^3\\+2d\end{matrix}+\begin{matrix}2bdx^4\\+cc\\+2c\end{matrix}\ \&c.$$

$$=y^2\times:1-2y^2+3y^4-4y^6+5y^8\ \&c.=y^2-2y^4+3y^6-4y^8+5y^{10}\ \&c.$$

Ex.

Ex. 11.

What is the ſquare root of $rr - zz + \frac{z^4}{3r^2} - \frac{2z^6}{45r^4} + \frac{z^8}{315r^6}$ — &c.

Here $z = 1$, $x = zz$, $a = rr$, $b = -1$, $c = \frac{1}{3r^2}$, $d = \frac{-2}{45r^4}$, $e = \frac{1}{315r^6}$ &c. and $m = \frac{1}{2}$, $\frac{m-1}{2} = -\frac{1}{4}$, $\frac{m-2}{3} = -\frac{1}{2}$, $\frac{m-3}{4} = -\frac{5}{8}$ &c. Then $\left.\overline{rr - zz + \frac{z^4}{3rr}\text{ \&c.}}\right|^{\frac{1}{2}} = r + \frac{1}{2} \times \frac{-1}{r}x - \frac{1}{8r^3}x^2 + \frac{1}{6r^3}x^2$ &c. $= r - \frac{zz}{2r} + \frac{z^4}{24r^3}$ &c.

Rather thus,

The quantity reduced is $rr \times : 1 - \frac{zz}{rr} + \frac{z^4}{3r^4} - \frac{2z^6}{45r^6}$ &c. Here $z = rr$, $a = 1$, $b = -\frac{1}{rr}$, $c = \frac{1}{3r^4}$, $d = \frac{-2}{45r^6}$ &c. Whence

$$\left.\overline{rr - zz + \frac{z^4}{3rr}\text{ \&c.}}\right|^{\frac{1}{2}} = r \times : 1 - \frac{x}{2rr} \left.\begin{matrix} - \frac{1}{8r^4} \\ + \frac{1}{6r^4} \end{matrix}\right\} x^2 \left.\begin{matrix} - \frac{1}{16r^6} \\ + \frac{1}{12r^6} \\ - \frac{1}{45r^6} \end{matrix}\right\} x^3 \text{ \&c.} = r \times : 1 - \frac{x}{2rr} + \frac{1}{24r^4}x^2 - \frac{1}{720r^6}x^3 + \text{\&c.}$$

$$= r - \frac{zz}{1.2r} + \frac{z^4}{1.2.3.4r^3} - \frac{z^6}{1.2.3.4.5.6.r^5} + \text{\&c.}$$

Ex. 12.

What is the ſquare root of

$$\frac{1}{rr - \frac{zz}{2} + \frac{z^4}{4rr} - \frac{z^6}{6r^4} + \frac{z^8}{8r^6} \;\&c.}$$

The quantity reduced is

$$\frac{1}{rr} \times \frac{1}{1 - \frac{zz}{2rr} + \frac{z^4}{4r^4} - \frac{z^6}{6r^6} \;\&c.}$$

Where $z = \frac{1}{rr}$, $x = zz$, $a = 1$, $b = -\frac{1}{2rr}$, $c = \frac{1}{4r^4}$, $d = \frac{-1}{6r^6}$, &c. and $m = -\frac{1}{2}$, $\frac{m-1}{2} = -\frac{3}{4}$, $\frac{m-2}{3} = -\frac{5}{6}$, $\frac{m-3}{4} = -\frac{7}{8}$ &c.

And $\sqrt{\frac{1}{rr - \frac{zz}{2} + \frac{z^4}{4rr} \;\&c.}} = \frac{1}{r} \times :$

$$\begin{array}{l} 1 + \frac{x}{4rr} + \frac{3x^2}{32r^4} + \frac{5x^4}{128r^6} \;\&c. \\ \qquad\qquad\quad - \frac{1}{8r^4} - \frac{3}{32r^6} \\ \qquad\qquad\qquad\qquad\;\; + \frac{1}{12r^6} \end{array}$$

$$= \frac{1}{r} + \frac{x}{4r^3} - \frac{x^2}{32r^5} + \frac{11x^3}{384r^7} - \&c.$$

SCHOLIUM.

From this problem the powers of a compound quantity are deduced as follows, which will be ſerviceable upon particular occaſions.

If $y = A + B + C + D$ &c. Then

$$y = A + B + C + D + E + F + G + H$$

$$\begin{array}{llllllll}
y^2 = A^2 & +2AB & +2AC & +2AD & +2AE & +2AF & +2AG & +2AH \text{ \&c.}\\
 & & +BB & +2BC & +2BD & +2BE & +2BF & +2BG\\
 & & & & +CC & +2CD & +2CE & +2CF\\
 & & & & & & +DD & +2DE
\end{array}$$

$$\begin{array}{lllllll}
y^3 = A^3 & +3A^2B & +3A^2C & +3A^2D & +3A^2E & +3A^2F & +3AAG \text{ \&c.}\\
 & & +3ABB & +6ABC & +6ABD & +6ABE & +6ABF\\
 & & & +B^3 & +3ACC & +6ACD & +6ACE\\
 & & & & +3BBC & +3BBD & +6BCD\\
 & & & & & +3BCC & +3ADD\\
 & & & & & & +3BBE\\
 & & & & & & +C^3
\end{array}$$

$$\begin{array}{llllll}
y^4 = A^4 & +4A^3B & +6A^2B^2 & +4AB^3 & +6A^2C^2 & +4B^3C \text{ \&c.}\\
 & & +4A^3C & +12A^2BC & +12AB^2C & +12ABC^2\\
 & & & +4A^3D & +12A^2BD & +12AB^2D\\
 & & & & +4A^3E & +12A^2CD\\
 & & & & +B^4 & +12A^2BE\\
 & & & & & +4A^3F
\end{array}$$

$$\begin{array}{llllll}
y^5 = A^5 & +5A^4B & +10A^3B^2 & +10A^2B^3 & +5AB^4 & +B^5 \text{ \&c.}\\
 & & +5A^4C & +20A^3BC & +30A^2B^2C & +20AB^3C\\
 & & & +5A^4D & +20A^3BD & +30A^2BC^2\\
 & & & & +10A^3CC & +30A^2B^2D\\
 & & & & +5A^4E & +20A^3CD\\
 & & & & & +20A^3BE\\
 & & & & & +5A^4F
\end{array}$$

$$\begin{array}{lllll}
y^6 = A^6 & +6A^5B & +15A^4B^2 & +20A^3B^3 & +15A^2B^4 \text{ \&c.}\\
 & & +6A^5C & +30A^4BC & +60A^3B^2C\\
 & & & +6A^5D & +30A^4BD\\
 & & & & +15A^4CC\\
 & & & & +6A^5E
\end{array}$$

$$\begin{array}{lllll}
y^7 = A^7 & +7A^6B & +21A^5B^2 & +35A^4B^3 & +35A^3B^4 \text{ \&c.}\\
 & & +7A^6C & +42A^5BC & +105A^4B^2C\\
 & & & +7A^6D & +42A^5BD\\
 & & & & +21A^5CC\\
 & & & & +7A^6E
\end{array}$$

$$\begin{array}{lllll}
y^8 = A^8 & +8A^7B & +28A^6B^2 & +56A^5B^3 & +70A^4B^4 \text{ \&c.}\\
 & & +8A^7C & +56A^6BC & +168A^5B^2C\\
 & & & +8A^7D & +56A^6BD\\
 & & & & +28A^6CC\\
 & & & & +8A^7E
\end{array}$$

$y^9 = A^9$

$$\begin{array}{llllll}
y^9 = A^9 & +9A^8B & +36A^7B^2 & +84A^6B^3 & +126A^5B^4 & \&c. \\
 & & +9A^8C & +72A^7BC & +252A^6B^2C & \\
 & & & +9A^8D & +72A^7BD & \\
 & & & & +36A^7CC & \\
 & & & & +8A^8E & \\
\hline
y^{10} = A^{10} & +10A^9B & +45A^8B^2 & +120A^7B^3 & +210A^6B^4 & \&c. \\
 & & +10A^9C & +90A^8BC & +360A^7B^2C & \\
\&c. & & & +10A^9D & +90A^8BD & \\
 & & & & +45A^8CC & \\
 & & & & +10A^9E &
\end{array}$$

In making uſe of any of theſe forms, the terms of the given ſeries muſt be ranged in order (Prob. xlviii.), and the whole terms thereof ſubſtituted one by one, in the room of the quantities A, B, C, D, &c. (Prob. xlix).

Ex. 1.

Let $a + bx + cx^2 + dx^3 + ex^4$ &c. *be cubed.*

$$A + B + C + D + E \ \&c.$$
$$= a + bx + cx^2 + dx^3 + ex^4 \ \&c.$$

that is $A = a$, $B = bx$, &c. Then

$$\begin{array}{lll}
(y^3)\ A^3 + 3A^2B & + 3A^2C \ \&c. & = \\
 & +3AB^2 &
\end{array}$$

$$\begin{array}{llllll}
a^3 & +3aabx & +3aacx^2 & +3aadx^3 & +3aaex^4 & \&c. \\
 & & +3abbx^2 & +6abcx^3 & +6abdx^4 & \\
 & & & +b^3x^3 & +3accx^4 & \\
 & & & & +3bbcx^4 &
\end{array}$$

Ex. 2.

What is the fourth power of

$$x - \frac{2}{x} + \frac{p}{x^3} - \frac{2cd}{x^5}.$$

$$A + B + C + D$$

$$= x - \frac{2}{x} + \frac{p}{x^3} - \frac{2cd}{x^5}$$

Then $y^4 = x^4 - 4x^3 \frac{2}{x} + 6xx \times \frac{4}{xx} - 4x \times \frac{8}{x^3}$ &c. $=$

$$+ 4x^3 \times \frac{p}{x^3} - 12xx \times \frac{2p}{x}$$

$$- 4x^3 \times \frac{2cd}{x^5}$$

$$x^4 - 8x^2 + 24 + 4p - \frac{32 + 24p + 8cd}{xx} \text{ \&c.}$$

Ex. 3.

Involve $2x^{\frac{1}{2}} + 3x^{\frac{5}{2}} - 4x^{\frac{7}{2}} + 5x^{\frac{9}{2}} - 6x^{\frac{11}{2}}$ &c. *to the 5th power.*

$$\text{A} + \text{B} + \text{C} + \text{D} + \text{E} \;\; \&c. = y.$$

$$= 2x^{\frac{1}{2}} + 3x^{\frac{5}{2}} - 4x^{\frac{7}{2}} + 5x^{\frac{9}{2}} - 6x^{\frac{11}{2}} \text{ \&c.} = y.$$

$$y^5 = 32x^{\frac{5}{2}} + 80x^{\frac{4}{2}} \times 3x^{\frac{5}{2}} + 80x^{\frac{3}{2}} \times 9x^{\frac{10}{2}}$$

$$+ 80x^{\frac{4}{2}} \times 4x^{\frac{7}{2}}$$

$$- 160x^{\frac{3}{2}} \times 12x^{\frac{12}{2}} -$$

$$+ 80x^{\frac{4}{2}} \times 5x^{\frac{9}{2}} \qquad \text{\&c.}$$

$$80x^{\frac{4}{2}} \times 6x^{\frac{11}{2}} \text{ \&c.} = 32x^{\frac{5}{2}} + 240x^{\frac{9}{2}} + 720x^{\frac{13}{2}} -$$

$$- 320x^{\frac{11}{2}} +$$

$$1920x^{\frac{15}{2}} - 480x^{15} \text{ \&c. that is}$$

$$420x^{13} \text{ \&c.}$$

$$y^5 = 32x^{\frac{5}{2}} + 240x^{\frac{9}{2}} - 320x^{\frac{11}{2}} + 720x^{13} - 1920x^{\frac{15}{2}}$$

$$+ 400x^{\frac{13}{2}} - 480x^{\frac{15}{2}}$$

$$\text{\&c.}$$

Or $y^5 = 32x^{\frac{5}{2}} + 240x^{\frac{9}{2}} - 320x^{\frac{11}{2}} + 1120x^{\frac{13}{2}} - 2400x^{\frac{15}{2}}$ &c. Here I omit all theſe terms, where I ſee the index of x exceeds $\frac{15}{2}$.

Or thus,

$$\begin{aligned}
&A + B + C + D + E + F \ \&c.\\
&= 2x^{\frac{1}{2}} + 0 + 3x^{\frac{5}{2}} - 4x^{\frac{7}{2}} + 5x^{\frac{9}{2}} - 6x^{\frac{11}{2}} \ \&c. \text{ then}
\end{aligned}$$

$$\begin{aligned}
y^5 = {} & 32x^{\frac{5}{2}} + 0 + 80x^{\frac{4}{2}}\\
& - 80x^{\frac{4}{2}} \times 4x^{\frac{7}{2}} + 80x^{\frac{3}{2}} \times 9x^{\frac{10}{2}} - 160x^{\frac{3}{2}} \times 12x^{\frac{12}{2}}\\
& \qquad\qquad + 80x^{\frac{4}{2}} \times 5x^{\frac{9}{2}} - 80x^{\frac{4}{2}} \times 6x^{\frac{11}{2}}\\
= {} & 32x^{\frac{5}{2}} + 240x^{\frac{7}{2}} - 320x^{\frac{11}{2}} + 1120x^{\frac{13}{2}} - 2400x^{\frac{15}{2}}
\end{aligned}$$

&c.

Ex. 4.

If $y = 1 + x^3 - 2x$, *what is* y^8.

$$\begin{aligned}
&A + B + C + D + E + F \ \&c.\\
&= 1 - 2x + x^3 + 0 + 0 \ \&c.
\end{aligned}$$

$$\begin{aligned}
y^8 = {} & 1 - 16x + 28 \times 4x^2 - 56 \times 8x^3 + 70 \times 16x^4 \ \&c.\\
& \qquad\qquad + 8x^3 - 56 \times 2x^4 + 168 \times 4x^5\\
& \qquad\qquad\qquad\qquad\qquad + 28\,x^6\\
= {} & 1 - 16x + 112x^2 - 448x^3 + 1120x^4 \ \&c.\\
& \qquad\qquad + 8x^3 - 112x^4 + 672x^5\\
& \qquad\qquad\qquad\qquad\qquad + 28x^6\\
= {} & 1 - 16x + 112x^2 + 8x^3 - 112x^4 \ \&c. \text{ that is,}\\
& \qquad\qquad - 448x^3 + 1120x^4
\end{aligned}$$

$y^8 = 1 - 16x + 112x^2 - 440x^3 - 1008x^4$ &c. This is ſuppoſing x to be very ſmall; but when x is very great, then x^3 muſt begin the ſeries;

Thus,

$$\begin{aligned}
&A + B + C + D + E + F \ \&c.\\
&= x^3 + 0 - 2x + 1 + 0 \ \&c. \quad \text{Then}
\end{aligned}$$

$y^3 = x^{24} + 0 - 8x^{21} \times 2x + 8x^{21} \times 1 + 28x^{18} \times 4xx$
or $y^3 = x^{24} - 16x^{22} + 8x^{21} + 112x^{20}$ &c.

PROBLEM LX.

To abridge an infinite ſeries, or denote it in a ſhort manner for working.

When a ſeries conſiſts of terms very much compounded, or having a great many factors; it is very laborious to reduce them into numbers. And when ſeveral factors in any term are contained in the ſucceeding terms; the work may be ſhortened, by making uſe of the preceding term or ſome part of it, inſtead of ſuch factors as are equivalent to it, in the following terms; as follows.

1 RULE.

Put A, B, C, D, *&c.* for the firſt, ſecond, third, fourth, *&c.* terms of the given ſeries. Then to get the coefficients thereof, divide every term by the preceding one, gives the coefficient of that term. Whence you will have a new ſeries equal to the former, and ſhorter deſignated.

Ex. 1.

$$\text{If } \overset{A}{z} + \overset{B}{\frac{z^3}{2a^2}} + \overset{C}{\frac{3z^5}{2.4a^4}} + \overset{D}{\frac{3.5z^7}{2.4.6a^6}} + \overset{E}{\frac{3.5.7z^9}{2.4.6.8a^8}} \&c = y.$$

$$\text{Then } z\Big)\frac{z^3}{2a^2}\Big(= \frac{zz}{2aa} = \text{coefficient of B} = \frac{B}{A},$$

$$\frac{z^3}{2a^2}\Big)\frac{3z^5}{2.4a^4}\Big(\frac{3z^2}{4a^2} = \text{coefficient of C} = \frac{C}{B},$$

$$\frac{3z^5}{2.4a^4}\Big)\frac{3.5z^7}{2.4.6a^6}\Big(\frac{5z^2}{6a^2} = \text{coefficient of D} = \frac{D}{C}, \&c.$$

Hence the ſeries becomes

$$z + \frac{zz}{2aa}A + \frac{3z^2}{4aa}B + \frac{5z^2}{6aa}C + \frac{7z^2}{8aa}D \,\&c. = y.$$

Ex.

Ex. 2.

Suppoſe $1 + \frac{v}{1.3} + \frac{v^2}{1.3.5} + \frac{v^3}{3.5.7} + \frac{v^4}{5.7.9} + \frac{v^5}{7.9.11}$ &c. $= y$.

Here $\frac{B}{A} = \frac{v}{3}$, $\frac{C}{B} = \frac{v}{5}$, $\frac{D}{C} = \frac{v}{7}$, $\frac{E}{D} = \frac{3v}{9}$, $\frac{F}{E} = \frac{5v}{11}$, &c.

Then the ſeries is,

$$1 + \frac{v}{3}A + \frac{v}{5}B + \frac{v}{7}C + \frac{3v}{9}D + \frac{5v}{11}E + \frac{7v}{13}F \text{ \&c.} = y.$$

Ex. 3.

Let $x - \frac{3x^2}{1.2} - \frac{5x^3}{1.2.3.4} - \frac{7x^4}{1.2.3.4.5.6}$ &c. *be given.*

$$x \Big) \frac{-3x^2}{1.2} \Big(\frac{-3x}{1.2}. \qquad \frac{-3x^2}{1.2} \Big) \frac{-5x^3}{1.2.3.4} \Big(\frac{5x}{3.3.4}.$$

$$\frac{-5x^3}{1.2.3.4} \Big) \frac{-7x^4}{1.2.3.4.5.6} \Big(\frac{7x}{5.5.6} \text{ \&c.}$$

And the ſeries is

$$x - \frac{3x}{2}A + \frac{5x}{3.3.4}B + \frac{7x}{5.5.6}C \text{ \&c.}$$

Or thus,

$$\frac{-3x^2}{2} \Big) \frac{-5x^3}{2.3.4} \Big(\frac{\frac{5}{3}x}{3.4}. \qquad \frac{-5x^3}{2.3.4} \Big) \frac{-7x^4}{2.3.4.5.6} \Big(\frac{\frac{7}{5}x}{5.6} \text{ \&c.}$$

And the ſeries

$$= x - \frac{3x}{2}\text{A} + \frac{\frac{5}{3}x}{3.4}\text{B} + \frac{\frac{7}{5}x}{5.6}\text{C}\ \&c.$$

Where A, B, C, &c. are the foregoing terms with their ſigns.

Ex. 4.

Suppoſe $bz - \frac{bz^3}{2.3aa} - \frac{bz^5}{5.2.4a^4} - \frac{bz^7}{7.2.4.6a^6} - \frac{bz^9}{9.2.4.6.8.a^8}$ &c. $= d$.

Then $bz\Big)\frac{-bz^3}{2.3aa}\Big(\frac{-zz}{2.3aa}$ = coefficient of B,

$\frac{-bz^3}{2.3aa}\Big)\frac{-bz^5}{5.2.4a^4}\Big(\frac{3zz}{4.5aa}$ = coefficient of C,

$\frac{-bz^5}{5.2.4a^4}\Big)\frac{-bz^7}{7.2.4.6a^6}\Big(\frac{5zz}{6.7aa}$ = coef. of D, &c.

And the ſeries is

$$bz - \frac{zz}{2.3aa}\text{A} + \frac{3z^2}{4.5aa}\text{B} + \frac{5zz}{6.7a^2}\text{C} + \frac{7zz}{8.9aa}\text{D}\ \&c.$$

2 RULE.

If there be ſome ſingle factor or factors, which are not in all the terms; ſet them aſide at preſent. Then put A, B, C, D, &c. for the remaining terms; and proceed as before. And at laſt reſtore theſe ſingle factors into their proper terms.

Ex.

Ex. 5.

If $x - \frac{3x^2}{1.2} - \frac{5x^3}{1.2.3.4} - \frac{7x^4}{1.2,3.4,5.6} - \frac{9x^5}{.2.3.4.5.6.7.8}$ &c. $= y$.

Here the factors 3, 5, 7, 9, *&c.* are not in ll the terms, and being left out, the ſeries is

$$x - \frac{x^2}{1.2} - \frac{x^3}{1.2.3.4} - \frac{x^4}{1.2.3.4.5.6} \text{ \&c.}$$

bridged to $x - \frac{x}{1.2}A + \frac{x}{3.4}B + \frac{x}{5.6}C + \frac{x}{7.8}D$ *&c.* and the factors reſtored, the ſeries becomes

$$x - \frac{x}{1.2}A \times 3 + \frac{x}{3.4}B \times 5 + \frac{x}{5.6}C \times 7 + \frac{x}{7.8}D \times 9 \text{ \&c.} = y.$$

Where A, B, C, *&c.* are he ſeveral terms with their proper ſigns; without he numbers, 3, 5, 7, *&c.*

Ex. 6.

If $bz - \frac{bz^3}{3.2aa} + \frac{bz^5}{5.2.4a^4} - \frac{bz^7}{7.2.4.6a^6} + \frac{bz^9}{.2.4.6.8a^8}$ &c. $= y$.

Then the factors 3, 5, 7, 9, *&c.* not being ommon to all the terms, are left out, and the eries is

$$bz - \frac{bz^3}{2aa} + \frac{bz^5}{2.4a^4} - \frac{bz^7}{2.4.6a^6} \text{ \&c.}$$

$$= bz - \frac{zz}{2aa}A - \frac{zz}{4aa}B - \frac{zz}{6aa}C - \frac{zz}{8aa}D \text{ \&c.}$$

And reſtoring the numbers, the ſeries will then be

$$\frac{bz}{1} - \frac{\frac{z^2}{2aa}A}{3} - \frac{\frac{zz}{4aa}B}{5} - \frac{\frac{zz}{6aa}C}{7} - \frac{\frac{zz}{8aa}D}{9}$$

&c. $= y$. Where A, B, C, *&c.* are the foregoing numerators, with their proper ſigns.

Ex. 7.

There is given

$$x - \frac{ax^3}{3.2} + \frac{bx^5}{5.2.4} - \frac{cx^7}{7.2.4.6} + \frac{dx^9}{9.2.4.6.8} \text{ \&c.}$$

curtailed, $x - \frac{x^3}{2} + \frac{x^5}{2.4} - \frac{x^7}{2.4.6} + \frac{x^9}{2.4.6.8}$ &c.

or ſhortened, $x - \frac{xx}{2}A - \frac{xx}{4}B - \frac{xx}{6}C - \frac{xx}{8}D$ &c.

compleat, $x - \frac{xx}{2}A \times \frac{a}{3} - \frac{xx}{4}B \times \frac{b}{5} - \frac{xx}{6}C \times \frac{c}{7} - \frac{xx}{8}D \times \frac{d}{9}$ &c.

Where A, B, C, *&c.* are the foregoing terms, excluſive of the following quantities.

Cor. 1. *If the firſt term of any transformed ſeries be multiplied by any number or quantity; the whole ſeries is multiplied thereby. For the firſt term is virtually contained in all the following terms. This is made plain by Ex.* 4.

Cor. 2. *In like manner* A, B, C *may be made to ſtand only for the coefficients, or otherwiſe, as any one pleaſes.*

PRO

PROBLEM LXI.

To find the finite value of an infinite ſeries, or what ſurd it is involved from.

RULE.

Divide all the terms by the firſt; then the firſt term will be 1. Then compare three terms of this ſeries with three terms of the ſeries Rule 2, Prob. lix. each with each, ſuppoſing a to be 1, and c, &c. 0; which two equations will find the index, and the ſecond term, if it is a binomial. If this does not ſucceed, compare four terms with four, for a trinomial; or five terms with five, for a quadrinomial; making $d = 0$, or $e = 0$, &c.

Ex. 1.

Suppoſe this ſeries $1 - \frac{y}{a} + \frac{y^2}{aa} - \frac{y^3}{a^3} + \frac{y^4}{a^4}$ &c.

Compare this with . . . $1 + mbx + m.\frac{m-1}{2}bbx^2$.

Then $mbx = -\frac{y}{x}$, and $m.\frac{m-1}{2}bbxx = \frac{yy}{aa}$, and dividing the laſt by the firſt, $\frac{m-1}{2}bxx = -\frac{y}{a} = mbx$; therefore $\frac{m-1}{2} = m$, and $2m = m - 1$, whence $m = -1$. Therefore $mbx = -bx = -\frac{y}{a}$, or $bx = \frac{y}{a}$. Whence the index is -1, and the ſecond term of the binomial (if it is one) is $\frac{y}{a}$. And the binomial $\overline{1 + \frac{y}{a}}\Big|^{-1}$, or $\frac{1}{1 + \frac{y}{a}}$ that is $\frac{a}{a+y}$ the root required; which ſucceeds.

Ex. 2.

Suppoſe $a + \frac{xx}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}$ &c.

Reduced $a \times : 1 + \frac{xx}{2aa} - \frac{x^4}{8a^4}$ &c.

Rule $1 + mbx + m . \frac{m-1}{2} bbxx.$

Here $mbx = \frac{xx}{2aa}$, and $m . \frac{m-1}{2} bbxx = - \frac{x^4}{8a^4}$, and by diviſion, $\frac{m-1}{2} bx = - \frac{xx}{4aa}$. Then $xx = mbx \times 2aa = - 4aa \times \frac{m-1}{2} bx$; whence $- m = m - 1$, and $2m = 1$, or $m = \frac{1}{2}$ the index. And $mbx = \frac{1}{2} bx = \frac{xx}{2aa}$, or $bx = \frac{xx}{aa}$ the ſecond term. And the ſurd is $a \times \overline{1 + \frac{xx}{aa}}\Big|^{\frac{1}{2}}$ or $\overline{aa + xx}\Big|^{\frac{1}{2}}$.

Ex. 3.

Let $x \sqrt[4]{8} - \frac{5aa}{8x} \sqrt[4]{8} - \frac{75a^4}{512x^3} \sqrt[4]{8} -$ *&c. be given.*

Reduced $x \sqrt[4]{8} \times : 1 - \frac{5aa}{8xx} - \frac{75a^4}{512x^4}$ &c.

Rule $1 + my + m . \frac{m-1}{2} yy$, putting $bx = y$.

Here $my = \frac{-5aa}{8xx}$, and $m . \frac{m-1}{2} yy = \frac{-75a^4}{512x^4}$, and by dividing $\frac{m-1}{2}y = \frac{15aa}{64xx}$; then $y = -\frac{5aa}{8xxm} = \frac{15aa}{32xx\,\overline{m-1}}$, and $-\frac{1}{m} = \frac{3}{4\times\overline{m-1}}$, or $-4m+4=3m$, and $7m=4$, whence $m=\frac{4}{7}$ the index. Alſo $y = \frac{-5aa}{8mxx} = -\frac{5aa}{\frac{32}{7}xx} = -\frac{35aa}{32xx}$ the ſecond term. And the binomial ſurd is

$$x\sqrt[4]{8}\times\overline{1-\frac{35aa}{32xx}}\Big|^{\frac{4}{7}}$$

Ex. 4.

Let the ſeries

$$a^{\frac{1}{2}} - \frac{y}{4a^{\frac{1}{2}}} + \frac{5y^2}{96a^{\frac{3}{2}}} + \frac{5y^3}{384a^{\frac{5}{2}}} + \frac{35y^4}{18432a^{\frac{7}{2}}}\ \&c.\ \textit{be}$$

propoſed.

This example reſolved like the foregoing, gives $m = -\frac{3}{2}$, and $\frac{y}{6a}$ for the ſecond term of the binomial. But $a^{\frac{1}{2}}\times\overline{1+\frac{y}{6a}}\Big|^{-\frac{3}{2}}$ does not produce the given ſeries. Whence we may conclude it has not a binomial root.

For a trinomial root; for brevity's ſake put 1, z, v for a, bx, cxx in the Rule, Prob. lix. which rule then becomes $\overline{1+z+v}|^{m} =$

$$1+mz+m.\frac{m-1}{2}zz+m.\frac{m-1}{2}.\frac{m-2}{3}z^3$$
$$+\ mv\ +\ m.\frac{m-1}{2}.2zv$$

and $a^{\frac{1}{2}} \times : 1 - \frac{y}{4a} + \frac{5y^2}{96aa} + \frac{5y^3}{384a^3}$ is the given ſeries reduced. Then we have theſe three equations, $mz = \frac{-y}{4a}$, $m \cdot \frac{m-1}{2} zz + mv = \frac{5yy}{96aa}$, and $m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} z^3 + m \cdot \frac{m-1}{2} . 2zv = \frac{5y^3}{384a^3}$. Divide the third by the firſt, and there comes out $\frac{m-1}{2} \cdot \frac{m-2}{3} zz + \overline{m-1} . v = \frac{-5yy}{96aa}$; add this to the ſecond, and we have $m \cdot \frac{m-1}{2} zz + \frac{m-1}{2} \cdot \frac{m-2}{3} zz + \overline{2m-1} . v = 0$, or $\frac{m-1}{2} \times \frac{4m-2}{3} zz + \overline{2m-1} . v = 0$. And ſquaring the firſt, $mmzz = \frac{yy}{16aa}$, and $\frac{yy}{aa} = 16mmzz$. Alſo $mv = \frac{5yy}{96aa} - m \cdot \frac{m-1}{2} zz = \frac{5}{96} \times 16mmzz - m \cdot \frac{m-1}{2} zz$. And $v = \frac{5}{6} mzz - \frac{m-1}{2} zz = \frac{2m+3}{6} zz$. Therefore $\frac{m-1}{2} \cdot \frac{4m-2}{3} zz + \overline{2m-1} . v = \frac{m-1}{3} . \overline{2m-1} . zz + \overline{2m-1} \cdot \frac{2m+3}{6} zz = 0$; or $\frac{2m-2+2m+3}{6} \times \overline{2m-1} = 0$, that is $\frac{4m+1}{6} \times \overline{2m-1} = 0$, and $\overline{4m+1} \times \overline{2m-1} = 0$. Which equation has two roots. $m = -\frac{1}{4}$, and $m = \frac{1}{2}$. If $m = -\frac{1}{4}$, then $z = \frac{-y}{4am} = \frac{-y}{-a} = \frac{y}{a}$, and $v = \frac{2m+3}{6} zz$ =

$= \frac{3 - \frac{1}{2}}{6} \times \frac{yy}{aa} = \frac{5yy}{12aa}$. And the ſurd root is $a^{\frac{1}{2}} \times \overline{1 + \frac{y}{a} + \frac{5yy}{12aa}}\Big|^{-\frac{1}{4}}$, which involved produces four terms of the ſeries but not the laſt.

And if $m = \frac{1}{2}$. Then $z = - \frac{7}{4am} = \frac{-y}{2a}$, and $v = \frac{2m + 3}{6} zz = \frac{4}{6} \times \frac{yy}{4aa} = \frac{yy}{6aa}$. And then the ſurd is $a^{\frac{1}{2}} \times : \overline{1 - \frac{y}{2a} + \frac{yy}{6aa}}\Big|^{\frac{1}{2}}$, which involved, produces all the terms of the given ſeries; and therefore is the root required.

PROBLEM LXII.

To revert an infinite ſeries; or to find the root of ſuch a ſeries.

1 RULE.

If the ſeries conſiſts of all the powers of z, as $Az + Bz^2 + Cz^3 + Dz^4 + Ez^5$ &c. $= y$; then ſubſtitute the values of the coefficients, A, B, C, D, *&c.* into the following form, for the root.

$$z = \frac{1}{A} y - \frac{B}{A^3} y^2 + \frac{2BB - AC}{A^5} y^3$$

$$+ \frac{5ABC - A^2D - 5B^3}{A^7} y^4$$

$$+ \frac{14B^4 - 21AB^2C + 6A^2BD + 3A^2C^2 - A^3E}{A^9} y^5$$

$$+ \frac{-42B^5 + 84AB^3C - 28A^2B^2D - 28A^2B^2C}{A^{11}}$$

$\underline{+ 7A^3BE + 7A^3DC - A^4F}\, y^6$ &c. This rule often diverges when y is great. See another Rule, Prob. XCIII.

For put $z = ay + by^2 + cy^3 + dy^4$ &c. Then

$$\begin{array}{rcl} zz &=& aay^2 + 2aby^3 + bby^4\ \&c. \\ && \qquad\qquad\qquad + 2ac \\ z^3 &=& \qquad\quad a^3y^3 + 3a^2by^4\ \&c. \\ z^4 &=& \qquad\qquad\qquad a^4y^4\ \&c. \end{array}$$

&c. Whence

$$\left.\begin{array}{rcl} Az &=& Aay + Aby^2 + Acy^3 + Ady^4\ \&c. \\ +Bz^2 &=& \qquad Ba^2y^2 + 2Baby^3 + Bbby^4\ \&c. \\ && \qquad\qquad\qquad\qquad + 2Bac \\ +Cz^3 &=& \qquad\qquad Ca^3y^3 + 3Ca^2by^4\ \&c. \\ +Dz^4 &=& \qquad\qquad\qquad\qquad Da^4y^4\ \&c. \end{array}\right\} = y.$$

&c.

Then making the homologous powers equal, $Aay = y$, and $a = \frac{1}{A}$. And $Ab + Ba^2 = 0$, or $b = \frac{-B}{A^3}$. Likewiſe $Ac + 2Bab + Ca^3 = 0$, and $c = \frac{2BB - AC}{A^5}$ In like manner $Ad + Bbb + 2Bac + 3Ca^2b + Da^4 = 0$, whence $d = \frac{5ABC - A^2D - 5B^3}{A^7}$; and ſo on.

Ex. 1.

Suppoſe $x - xx + x^3 - x^4 + x^5$ &c. $= y$, *to find the value of* x *in terms of* y.

Here $z = x$, $A = 1$, $B = -1$, $C = 1$, $D = -1$, *&c.* Whence $x = \frac{y}{1} + \frac{1}{1}y^2 + \frac{2-1}{1}y^3 + \frac{-5+1+5}{1}y^4$ &c. $= y + y^2 + y^3 + y^4 + y^5$ &c.

Ex. 2.

Let $z = x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}$ &c. *to find* x *in a ſeries of* z.

Here $z = x$, $y = z$, $A = 1$, $B = \frac{1}{2}$, $C = \frac{1}{3}$, $D =$

$D = \frac{1}{4}$, $E = \frac{1}{5}$, *&c.* and $x = \frac{1}{1}z + \frac{1}{2}z^2$

$$+ \frac{\frac{1}{2} - \frac{1}{3}}{1} z^3 + \frac{5 \times \frac{1}{6} - \frac{1}{4} - 5 \times \frac{1}{8}}{1} z^4 \text{ \&c.}$$

$$= z - \frac{1}{2} z^2 + \frac{1}{6} z^3 - \frac{1}{24} z^4 + \frac{1}{120} z^5 \text{ \&c.}$$

$$= z - \frac{zz}{1.2} + \frac{z^3}{2.3} - \frac{z^4}{2.3.4} + \frac{z^5}{2.3.4.5} \text{ \&c.}$$

that is, $x = z - \frac{z}{2}A - \frac{z}{3}B - \frac{z}{4}C - \frac{z}{5}D$ *&c.* where A, B, C, *&c.* are the foregoing terms, with their ſigns.

Ex. 3.

Suppoſe $r - \frac{aa}{2r} + \frac{a^4}{24r^3} - \frac{a^6}{720r^5} + \frac{a^8}{40320r^7}$ − &c. $= c$, *to find a.*

Put $r - c = v$. Then $\frac{aa}{2r} - \frac{a^4}{24r^3} + \frac{a^6}{720r^5} - \frac{a^8}{40320r^7}$ &c. $= v$. Here $z = aa$, $y = v$, $A = \frac{1}{2r}$, $B = \frac{-1}{24r^3}$, $C = \frac{1}{720r^5}$, $D = \frac{-1}{40320r^7}$, &c. Whence

$$aa = 2rv - \frac{\frac{-1}{24r^3}}{\frac{1}{8r^3}} vv + \frac{\frac{1}{288r^6} - \frac{1}{1440r^6}}{\frac{1}{32r^5}} v^3 \text{ \&c.}$$

$$= 2rv + \frac{1}{3} vv + \frac{4}{45r} v^3 + \frac{1}{35r^2} v^4 \text{ \&c.}$$

And extracting the root, $a = \sqrt{2rv} \times : 1 + \frac{v}{12r} + \frac{3v^2}{160rr} + \frac{5v^3}{896r^3} + \text{\&c.}$

2 RULE.

2 RULE.

If the ſeries conſiſts of the odd powers of z, as $Az + Bz^3 + Cz^5 + Dz^7$ &c. $= y$. Subſtitute the values of the coefficients A, B, C, &c. into the following form; which will give the root.

$$z = \frac{1}{A}y - \frac{B}{A^4}y^3 + \frac{3BB - AC}{A^7}y^5 + \frac{8ABC - A^2D - 12B^3}{A^{10}}y^7$$

$$+ \frac{55B^4 - 55AB^2C + 10A^2BD + 5A^2C^2 - A^3E}{A^{13}}y^9$$

&c.

For put $z = ay + by^3 + cy^5 + dy^7$ &c.

$$\begin{array}{rlllll}
\text{Then} & z^3 = & a^3y^3 & + 3a^2by^5 & + 3a^2cy^7 \\
& & & & + 3abb \\
& z^5 = & & a^5y^5 & + 5a^4by^7 \\
& z^7 = & & & a^7y^7 \\
& \text{&c.} & & &
\end{array}$$

$$\left.\begin{array}{rlllll}
\text{And } Az = & Aay & + Aby^3 & + Acy^5 & + Ady^7 \text{ &c.} \\
+ Bz^3 = & & + Ba^3y^3 & + 3Ba^2by^5 & + 3Ba^2cy^7 \\
& & & & + 3Babb \\
+ Cz^5 = & & & + Ca^5y^5 & + 5Ca^4by^7 \\
+ Dz^7 = & & & & + Da^7y^7
\end{array}\right\} = y$$

Then equating the coefficients of like terms; $Aa = 1$, $Ab + Ba^3 = 0$, $Ac + 3Ba^2b + Ca^5 = 0$, $Ad + 3Ba^2c + 3Babb + 5Ca^4b + Da^7 = 0$, &c. whence $a = \frac{1}{A}$, $b = -\frac{Ba^3}{A} = -\frac{B}{A^4}$. Likewiſe $c = \frac{3BB - AC}{A^7}$, $d = \frac{8ABC - A^2D - 12B^3}{A^{10}}$, &c.

Ex. 4.

Let $a - \frac{a^3}{2\cdot 3dd} + \frac{a^5}{2\cdot 3\cdot 4\cdot 5d^4} - \frac{a^7}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7d^6} + \&c.$ $= y$; *to find* a.

Here $z = a$, $y = y$, $A = 1$, $B = -\frac{1}{2\cdot 3dd}$, $C = \frac{1}{2\cdot 3\cdot 4\cdot 5d^4}$, $D = -\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7d^6}$, &c.

$$\text{Whence } a = y + \frac{1}{2\cdot 3dd}y^3 + \overline{\frac{1}{3\cdot 4d^4} - \frac{1}{2\cdot 3\cdot 4\cdot 5d^4}} \times y^5 + \overline{-\frac{1}{2\cdot 3\cdot 3\cdot 5d^6} + \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7d^6} + \frac{1}{2\cdot 3\cdot 3d^6}} \times y^7$$

$$\&c. = y + \frac{1}{2\cdot 3dd}y^3 + \frac{3}{2\cdot 4\cdot 5d^4}y^5 + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot d^5}y^7 + \&c.$$

Ex. 5.

Suppose $y + \frac{y^3}{2\cdot 3dd} + \frac{3y^5}{2\cdot 4\cdot 5d^4} + \frac{3\cdot 5y^7}{2\cdot 4\cdot 6\cdot 7d^6} +$ &c. $= a$, *to find* y.

Here $z = y$, $y = a$, $A = 1$, $B = \frac{1}{2\cdot 3dd}$, $C = \frac{3}{2\cdot 4\cdot 5d^4}$, $D = \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 7d^6}$, &c. Then

$$y = a - \frac{1}{2\cdot 3dd}a^3 + \overline{\frac{1}{2\cdot 2\cdot 3d^4} - \frac{3}{2\cdot 4\cdot 5d^4}} \times a^5 + \overline{\frac{1}{2\cdot 5} - \frac{5}{2\cdot 4\cdot 2\cdot 7} - \frac{1}{2\cdot 3\cdot 3}} \times \frac{a^7}{d^6}, \&c. = a - \frac{1}{2\cdot 3dd}a^3 + \frac{1}{2\cdot 3\cdot 4\cdot 5d^4}a^5 - \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7d^6}a^7 + \&c.$$

Ex. 6.

Given $bz - \frac{bz^3}{6aa} - \frac{bz^5}{40a^4} - \frac{bz^7}{336a^6} - \frac{bz^9}{3456a^8}$ &c. $= d$, *to find* z.

Dividing by b, $z - \frac{z^3}{6aa} - \frac{z^5}{40a^4} - \frac{z^7}{336a^6}$ &c. $= \frac{d}{b} = n$. Then $y = n$, A = 1, B $= -\frac{1}{6aa}$, C $= -\frac{1}{40a^4}$; D $= -\frac{1}{336a^6}$, &c. then will

$$z = n + \frac{n^3}{6aa} + \overline{\frac{1}{12a^4} + \frac{1}{40a^4}} \times n^5 + \overline{\frac{8}{6.40} + \frac{1}{336} + \frac{12}{6.6.6}} \times \frac{n^7}{a^6} + \text{\&c.} = n + \frac{n^3}{6aa} + \frac{13}{120a^4} n^5 + \frac{463}{840} n^7 + \text{\&c.}$$

3 RULE.

When the ſeries conſiſts of any powers of z denoted by m and n, as $Az^m + Bz^{m+n} + Cz^{m+2n} + Dz^{m+3n} + Ez^{m+4n}$ &c. $= y$. Then ſubſtitute the values of the coefficients, A B, C, *&c.* into this form, for the root or value of z.

Put $v = \frac{y}{A}$. Then

$$z = v^{\frac{1}{m}} - \frac{B}{mA} v^{\frac{1+n}{m}} + \frac{\overline{m+1+2n}.BB - 2mAC}{2mmAA} v^{\frac{1+2n}{m}}$$

— 2mm

$$\left.\begin{array}{r} -\dfrac{2mm+9mn+9nn+3m+6n+1}{6m^3A^3}B^3 \\ +\dfrac{m+3n+1}{mmA^2}BC \\ -\dfrac{D}{mA} \\ \&c. \end{array}\right\} v^{\frac{1+3n}{m}}$$

For put $z = v^{\frac{1}{m}} + bv^{\frac{1+n}{m}} + cv^{\frac{1+2n}{m}} + dv^{\frac{1+3n}{m}} + \&c.$

Then dividing the given ſeries by A, we have $z^m + \frac{B}{A}z^{m+n} + \frac{C}{A}z^{m+2n}$ &c. $= \frac{y}{A} = v$.

Whence by involution,

$$\left.\begin{array}{l} z^m = v + mbv^{\frac{m+n}{m}} + \begin{array}{r} mcv^{\frac{m+2n}{m}} \\ + m\cdot\dfrac{m-1}{2}bb \end{array} \&c. \\ \dfrac{B}{A}z^{m+n} = \dfrac{B}{A}\times : v^{\frac{m+n}{m}} + \overline{m+n}.\,bv^{\frac{m+2n}{m}} \&c. \\ \dfrac{C}{A}z^{m+2n} = \dfrac{C}{A}\times : \qquad v^{\frac{m+2n}{m}} \&c. \end{array}\right\} = v.$$

$$\frac{D}{A}z^{m+3n} = \&c.$$

Then equating the coefficients, $mb + \frac{B}{A} = 0$, and $b = \frac{-B}{mA}$. And $mc + m\cdot\frac{m-1}{2}bb + \overline{m+n}.\frac{bB}{A} + \frac{C}{A} = 0$, and $c = \frac{\overline{m+1+2n}.BB - 2mAC}{2m^2A^2}$ &c.

Note, in all theſe rules, I have only purſued theſe ſeries to a few terms; to have gone farther would have taken up too much room: but the method is viſible.

Ex. 7.

Suppoſe $\frac{1}{2}xx + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5$ &c. $= y$.

Here $z = x$, $v = 2y$, $A = \frac{1}{2}$, $B = \frac{1}{3}$, $C = \frac{1}{4}$, $D = \frac{1}{5}$, *&c.* and $m = 2$, $n = 1$. Whence

$$x = v^{\frac{1}{2}} - \frac{1}{3}v^{\frac{2}{2}} + \frac{5BB - 4AC}{8AA}v^{\frac{3}{2}} \text{ \&c.} = v^{\frac{1}{2}} - \frac{1}{3}v + \frac{1}{2\cdot 18}v^{\frac{3}{2}} + \frac{1}{270}v^2 \text{ \&c.}$$

Ex. 8.

Let $x - \frac{a^2}{2x} + \frac{a^4}{6x^3} - \frac{a^6}{24x^5} +$ &c. $= y = v$.

Here $z = x$, $m = 1$, $n = -2$, $A = 1$, $B = -\frac{aa}{2}$, $C = \frac{a^4}{6}$, $D = -\frac{a^6}{24}$ &c. and

$$x = y^{\frac{1}{1}} + \frac{aa}{2}y^{-\frac{1}{1}} + \frac{-2BB - 2AC}{2A^2}y^{-3} \text{ \&c.} = y + \frac{aa}{2y} - \frac{5a^4}{12y^3} + \frac{5a^6}{8y^5} \text{ \&c.}$$

Ex. 9.

Let $x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{8}x^{\frac{3}{2}} - \frac{1}{16}x^{\frac{5}{2}} - \frac{5}{128}x^{\frac{7}{9}}$ &c. $= z$, *to find* x.

In this Ex. $z = x$, $v = z$, $m = -\frac{1}{2}$, $n = 1$, $A = 1$, $B = -\frac{1}{2}$, $C = -\frac{1}{8}$, $D = -\frac{1}{16}$, $E = -\frac{5}{128}$ &c.

Whence $x = z^{-2} - z^{-4} + \frac{2\frac{1}{2}BB + C}{\frac{1}{2}} z^{-6} + \overline{14B^3 + 14BC + 2D} \times z^{-8}$ &c. that is

$x = \frac{1}{zz} - \frac{1}{z^4} + \frac{1}{z^6} - \frac{1}{z^8}$ &c.

Cor. 1. *If you would find any power of y; find y in a ſeries of z, and then involve that ſeries to the power required, or elſe put* $s = y^r$; *then find s.* (y^r) *from ſuch a ſeries as this,*

$$As^{\frac{m}{r}} + Bs^{\frac{m+n}{r}} + Cs^{\frac{m+2n}{r}} \text{ \&c.} = y;$$

by the laſt rule.

Cor. 2. *The reverted ſeries is of the ſame form as the given ſeries; for otherwiſe they are not convertible into one another.*

PROBLEM. LXIII.

To extract the root of a ſeries containing all the powers of two letters.

1 RULE.

If the ſeries conſiſts of all the ſingle powers of z and y, as $az + bz^2 + cz^3 + dz^4$ &c. $= gy + hy^2 + jy^3 + ky^4$ &c. ſubſtitute the values of the coefficients in the following form, for the root.

$$z = \frac{g}{a}y + \frac{h - bA^2}{a}y^2 + \frac{j - 2bAB - cA^3}{a}y^3$$

$$+ \frac{k - bB^2 - 2bAC - 3cA^2B - dA^4}{a}y^4$$

$$+ \frac{l - 2bBC - 2bAD - 3cAB^2 - 3cA^2C - 4dA^3B - eA^5}{a}y^5$$

$$+ \frac{m - 2bBD - bC^2 - 2bAE - cB^3 - 6cABC - 3cA^2D - 6dA^2B^2 - 4dA^3C - 5eA^4B - fA^6}{a}y^6 \text{ \&c.}$$

Where A, B, C, *&c.* are the coefficients of the firſt, ſecond, third, *&c.* terms.

Let $z = Ay + By^2 + Cy^3 + Dy^4$ &c. Then

$$\left.\begin{array}{llllll} az & = aAy & + aBy^2 & + aCy^3 & + aDy^4 & \&c. \\ + bz^2 & = & + bA^2y^2 & + 2bABy^3 & + bBBy^4 & \\ & & & & + 2bAC & \\ + cz^3 & = & & + cA^3y^3 & + 3cA^2By^4 & \\ + dz^4 & = & & & + dA^4y^4 & \\ \&c. & & & & & \end{array}\right\}$$

$$= gy + hy^2 + jy^3 + ky^4 \ \&c.$$

And equating the coefficients, $aA = g$, and $A = \frac{g}{a}$. Alſo $aB + bA^2 = h$, and $B = \frac{h - bAA}{a}$. Alſo $aC + 2bAB + cA^3 = j$, and $C = \frac{j - 2bAB - cA^3}{a}$. Again $aD + bB^2 + 2bAC + 3cA^2B + dA^4 = k$, and $D = \frac{k - bB^2 - 2bAC - 3cA^2B - dA^4}{a}$. *&c.*

Ex. 1.

Let $x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24}$ &c. $= \frac{1}{2}y + \frac{1}{3}y^2 + \frac{1}{4}y^3 + \frac{1}{5}y^4$ &c. *to find* x.

Here $z = x$, $y = y$, $a = 1$, $b = -\frac{1}{2}$, $c = \frac{1}{6}$, $d = -\frac{1}{24}$, &c. and $g = \frac{1}{2}$, $h = \frac{1}{3}$, $j = \frac{1}{4}$, $k = \frac{1}{5}$, &c.

Then

$$x = \frac{1}{2}y + \frac{\frac{1}{3} + \frac{1}{8}}{1}y^2 + \frac{\frac{1}{4} + \frac{11}{48} - \frac{1}{48}}{1}y^3 \text{ \&c. or}$$

$$x = \frac{1}{2}y + \frac{11}{24}y^2 + \frac{11}{24}y^3 + \frac{1381}{2880}y^4 \text{ \&c.}$$

Ex. 2.

Suppoſe $z + \frac{z^3}{6dd} + \frac{3z^5}{40d^4} + \frac{5z^7}{112d^6}$ &c. $= ny + \frac{ny^3}{6dd} + \frac{3ny^5}{40d^4} + \frac{5ny^7}{112d^6}$ &c. *to find* z.

Comparing this with the rule, and we have $a = 1$, $b = 0$, $c = \frac{1}{6dd}$, $d = 0$, $e = \frac{3}{40d^4}$, $f = 0$, &c. $g = n$, $h = 0$, $j = \frac{n}{6dd}$, $k = 0$, $l = \frac{3n}{40d^4}$, $m = 0$, &c. Whence

$$z = \frac{n}{1}y + 0y^2 + \overline{\frac{n}{6dd} - \frac{A^3}{6dd}} \times y^3 + 0y^4 + \overline{\frac{3n}{40d^4} - \frac{AAC}{2dd} - \frac{3A^5}{40d^4}} \times y^5 \text{ \&c.}$$

where B, D, *&c.* $= 0$; that is

$$z = ny + \frac{n - n^3}{6dd}y^3 + \overline{\frac{3n}{40d^4} - \frac{n^3}{12d^4} + \frac{n^5}{120d^4}} \times y^5 \text{ \&c.} = ny + \frac{n - n^3}{6dd}y^3 + \frac{9n - 10n^3 + n^5}{120d^4}y^5 = ny + \frac{n}{1} \times \frac{1 - nn}{6dd}y^3 + \frac{n - n^3}{6dd} \times \frac{9 - nn}{20dd}y^5 \text{ \&c.}$$

Or $z = ny + \frac{1 - nn}{2.3dd}yy\mathrm{A} + \frac{9 - nn}{4.5dd}yy\mathrm{B}$ &c. where A, B, *&c.* are the foregoing terms.

2 RULE.

In two ſeries conſiſting of the powers and products of z and y; as
$az+bz^2+cz^3+dz^4$ &c. $+fy+gzy+hz^2y+jz^3y$ &c. $+ly^2+my^2z+ny^2z^2$ &c. $+py^3+qy^3z$ &c. $+sy^4$ &c. $=0$.
Then ſubſtitute the values of the coefficients, into the following form;

$$z=-\frac{f}{a}y-\frac{l+gA+hA^2}{a}y^2-\frac{2bAB+cA^3+p+gB+mA+hA^2}{a}y^3$$
$$-\frac{2bAC+bBB+3cA^2B+dA^4+s+gC+mB+qA+2hAB+nA^2+jA^3}{a}y^4 \text{ \&c.}$$

Where A, B, C, *&c.* are the coefficients of the firſt, ſecond, third, *&c.* terms.

For put $z=Ay+By^2+Cy^3+Dy^4$ &c.
Then

$$\left.\begin{array}{lclll}
az & = & aAy & +\ aBy^2 & +\ aCy^3 \text{ \&c.} \\
+\,bz^2 & = & & bA^2y^2 & +\ 2bABy^3 \\
+\,cz^3 & = & & & cA^3y^3 \\
+\,fy & = & fy & & \\
+\,ly^2 & = & & +\ ly^2 & \\
+\,py^3 & = & & & +\ py^3 \\
+\,gyz & = & & +\ gAy^2 & +\ gBy^3 \\
+\,my^2z & = & & & +\ mAy^3 \\
+\,hyz^2 & = & & & +\ hA^2y^3 \\
\text{\&c.} & & & &
\end{array}\right\}=0.$$

Then equating the coefficients, $aA+f=0$, $aB+bA^2+l+gA=0$, *&c.* whence $A=-\frac{f}{a}$, $B=-\frac{bA^2+l+gA}{a}$, $C=-\frac{2bAB+cA^3+p+gB+mA+hA^2}{a}$, &c.

Ex. 3.

Suppoſe $2y + \frac{1}{12}y^3 + \frac{1}{2}x - \frac{1}{4}xx + xy - \frac{3}{8}xyy + x^2y^2 = 0$; *to find* y.

Here $z = y$, $y = x$, $a = 2$, $c = \frac{1}{12}$, $f = \frac{1}{2}$, $g = 1$, $h = -\frac{3}{8}$, $l = -\frac{1}{4}$, $n = 1$; b, d, j, m, p, q, $s = 0$. Therefore

$$y = -\frac{1}{4}x - \frac{-\frac{1}{4} + A}{2}x^2 - \frac{\frac{1}{12}A^3 + B - \frac{3}{8}AA}{2}x^3 - \frac{\frac{1}{4}A^2B + C - \frac{3}{4}AB + A^2}{2}x^4 \text{ \&c.} = -\frac{1}{4}x + \frac{1}{4}x^2 - \frac{173}{768}x^3 + \frac{43}{384}x^4 \text{ \&c.}$$

PROBLEM LXIV.

To extract the root of an adfected equation, by a ſeries.

1 RULE.

If the equation conſiſts of terms which contain the powers of x and y; and you want the value of y, in a ſeries of x. Make the equation $= 0$, and aſſume an indetermined ſeries for the root, as $y = Ax^n + Bx^{n+r} + Cx^{n+s} + Dx^{n+t}$ &c. wherein the indices $n + r$, $n + s$, &c. continually increaſe if x be very ſmall; but they decreaſe if x be great; the firſt is an *aſcending* ſeries, and the latter a *deſcending* one. By this means the ſeries will converge; every following term growing ſtill leſs, till they vaniſh or become of no moment.

For y and its powers in the given equation, ſubſtitute the firſt term Ax^n and its powers. Then to determine n, put the two leaſt indices equal to each other, for an aſcending ſeries; or the two greateſt, for a deſcending one. And if it appears not at firſt ſight, which is the two leaſt, or two greateſt; it will be known, by comparing every two of the indexes.

Then to determine r, s, t, &c. ſubſtitute its value for n, in all theſe indices, and having taken the leaſt for an aſcending ſeries, or the greateſt for a deſcending one; ſubtract it from each of the reſt. Then take theſe remainders, and add them to themſelves and to one another, all poſſible ways; and theſe remainders, and the ſums reſulting, taken in order, will be the values of r, s, t, &c. which will be affirmative, in an aſcending ſeries; but negative in a deſcending one. Then put theſe values in the ſeries, $Ax^n + Bx^{n+r} + Cx^{n+2r}$ &c.

Then to find the coefficients A, B, C, D, *&c.* ſubſtitute the laſt ſeries for the powers of y, in the equation; and put the coefficient of each power of x, ſucceſſively $= 0$; and A, B, C, *&c.* will be gradually found from theſe equations.

Ex. 1.

Let $a^4x^2 - a^4xy + x^6 = ay^5$, *to find* y.

By reduction $a^4x^2 - a^4xy + x^6 - ay^5 = 0$. Put $y = Ax^n + Bx^{n+r} + Cx^{n+s} + Dx^{n+t}$ &c. ſubſtitute Ax_n for y, in the equation, and we have $a^4x^2 - a^4Ax^{n+1} + x^6 - aA^5x^{5n} = 0$. Then equating the indices, $n + 1 = 2$, for the leaſt, or $5n = 6$ for the greateſt indices.

For an aſcending ſeries.

Here $n + 1 = 2$, and $n = 1$. Then the indices 2, $n + 1$, 6, $5n$, become 2, 2, 6, 5. Subtract 2 from the reſt, and you have 3, 4; out of which is compoſed this ſeries 3, 4, 6, 7, 8, 9, 10, &c. for the values of r, s, t, &c. whence the form of the ſeries will be $y = Ax + Bx^4 + Cx^5 + Dx^7 + Ex^8$ &c. This ſeries ſubſtituted for y in the given equation, will be as follows:

$$\begin{array}{lclllll}
a^4x^2 & = & a^4x^2 & & & & \\
-a^4xy & = & -a^4Ax^2 & -a^4Bx^5 & -a^4Cx^6 & -a^4Dx^8 & \&c. \\
+x^6 & = & * & * & +x^6 & & \\
-ay^5 & = & * & -aA^5x^5 & * & -5aA^4Bx^8 & \&c. \\
\hline
0 & = & 0 & 0 & 0 & 0 &
\end{array}$$

Then equating the homologous terms $a^4 - a^4A = 0$, and $A = 1$. Alſo $-a^4B - aA^5 = 0$, and $B = \frac{-A^5}{a^3} = -\frac{1}{a^3}$. Again, $-a^4C + 1 = 0$, and $C = \frac{1}{a^4}$. Likewiſe $-a^4D - 5aA^4B = 0$. Whence $D = +\frac{5}{a^6}$, &c. Then the ſeries or root required is

$$y = x - \frac{x^4}{a^3} + \frac{x^5}{a^4} + \frac{5x^7}{a^6} \ \&c.$$

For a deſcending ſeries.

Here $5n = 6$, the greateſt indices, and $n = 1\frac{1}{5}$, and ſubſtituting this value of n, the indices 2, $n + 1$, 6, $5n$ become 2, $2\frac{1}{5}$, 6, 6, and the remainders $-3\frac{4}{5}$, -4; and r, s, t, &c. will be $-3\frac{4}{5}$, -4, $-7\frac{3}{5}$, $-7\frac{4}{5}$, -8, &c. and the ſeries becomes $y = Ax^{1\frac{1}{5}} + Bx^{-2\frac{3}{5}} + Cx^{-2\frac{4}{5}} + Dx^{-6\frac{2}{5}}$ &c. which ſubſtituted in the given equation, will be

ax^2

$$\begin{array}{rclllll}
a^4x^2 & = & * & * & & + a^4x^2 & \\
-a^4xy & = & * & -a^4Ax^{2\frac{1}{5}} & * & & -a^4Bx^{-1\frac{3}{5}} \\
+x^6 & = & +x^6 & & & & \&c. \\
-ay^5 & = & -aA^5x^6 & -5aA^4Bx^{2\frac{1}{5}} & & -5aA^4Cx^2 & -5aA^4Dx^{-1\frac{3}{5}} \\
 & & & & & & -10aA^3B^2 \\
 & = & 0. & & & & \&c.
\end{array}$$

Then equating the coefficients of like terms, $1 - aA^5 = 0$, and $A = \frac{1}{a^{\frac{1}{5}}}$. Likewiſe $-a^4A - 5aA^4B = 0$, and $B = -\frac{1}{5}a^{3\frac{3}{5}}$, alſo $a^4 - 5aA^4C = 0$, and $C = \frac{1}{5}a^{3\frac{4}{5}}$. Alſo $-a^4B - 5aA^4D - 10aA^3B^2 = 0$, and $D = -\frac{1}{25}a^{7\frac{2}{5}}$ &c. Whence the root is

$$y = \frac{x^{1\frac{1}{5}}}{a^{\frac{1}{5}}} - \frac{a^{3\frac{3}{5}}}{5x^{2\frac{3}{5}}} + \frac{a^{3\frac{4}{5}}}{5x^{2\frac{4}{5}}} - \frac{a^{7\frac{2}{5}}}{25x^{6\frac{2}{5}}} \ \&c.$$

If you put $n + 1 = 6$, the indices will be 2, 6, 6, 25; but 6 is neither the greateſt nor the leaſt, therefore this ſucceeds not.

If you put $5n = 2$, the indices will be 2, $1\frac{2}{5}$, 6, 2; but here alſo 2 is neither the greateſt nor the leaſt. Therefore this will not ſucceed.

If we put $n + 1 = 5n$, the indices will be 2, $1\frac{1}{4}$, 6, $1\frac{1}{4}$; and $1\frac{1}{4}$ being the leaſt, this will do for an aſcending ſeries; and the form of it will be $y = Ax^{\frac{1}{4}} + Bx + Cx^{1\frac{3}{4}} + Dx^{2\frac{1}{2}}$ &c.

Ex. 2.

Let $a^3x + ax^3 - a^3y - y^4 = 0$, *be propoſed.*

Putting Ax^n for y, the equation becomes $a^3x + ax^3 - a^3Ax^n - A^4x^{4n} = 0$. Then put $n = 1$

for the leaſt indexes, then the indexes become 1, 3, 1, 4; and the differences 2, 3; and r, s, t, &c. 2, 3, 4, 5, 6, *&c.* and the ſeries $Ax + Bx^3 + Cx^4 + Dx^5$ &c. $= y$. Whence

$$\begin{array}{rcllllll}
a^3x & = & a^3x & & & & & \\
+ax^3 & = & & +ax^3 & & & & \\
-a^3y & = & -a^3Ax & -a^3Bx^3 & -a^3Cx^4 & -a^3Dx^5 & -a^3Ex^6 & \\
 & & & & & & \&c. & \\
-y^4 & = & & & -a^4x^4 & & -4a^3Bx^6 & \\
 & & & & & & \&c. &
\end{array}$$

Then equating the coefficients, $a^3A = a^3$, and $A = 1$. In like manner $B = \frac{1}{aa}$, $C = -\frac{1}{a^3}$, $D = 0$, $E = -\frac{4}{a^5}$ &c. and the root is

$$y = x + \frac{1}{aa}x^3 - \frac{1}{a^3}x^4 - \frac{4}{a^5}x^6 \text{ &c.}$$

Otherwiſe for a deſcending ſeries.

Put $3 = 4n$, then $n = \frac{3}{4}$, and the indices are 1, 3, $\frac{3}{4}$, 3; and the differences -2, $-2\frac{1}{4}$, and r, s, t, &c. $= -2$, $-2\frac{1}{4}$, -4, $-4\frac{1}{4}$, *&c.* and $y = Ax^{\frac{3}{4}} + Bx^{-1\frac{1}{4}} + Cx^{-1\frac{1}{2}} + Dx^{-3\frac{1}{4}}$ &c.

Whence

$$\begin{array}{rcllll}
a^3x & = & * & +a^3x & & \\
+ax^3 & = & ax^3 & * & & \\
-a^3y & = & * & * & -a^3Ax^{\frac{3}{4}} & * \\
-y^4 & = & -A^4x^3 & -4A^3Bx & -4A^3Cx^{\frac{3}{4}} & -4A^3Dx^{-1} \\
 & & & & & -6A^2B^2
\end{array}$$

Then by equating the coefficients, $A^4 = a$, and $A = a^{\frac{1}{4}}$; alſo $B = \frac{a^{2\frac{1}{4}}}{4}$, $C = -\frac{a^{2\frac{1}{2}}}{4}$, $D = -\frac{3a^{4\frac{1}{4}}}{32}$, &c.

And

$$y = a^{\frac{1}{4}} x^{\frac{3}{4}} + \frac{a^{2\frac{1}{4}}}{4} x^{-1\frac{1}{4}} - \frac{a^{2\frac{1}{2}}}{4} x^{-1\frac{1}{2}} -$$

$$\frac{3a^{4\frac{1}{4}}}{32} x^{-3\frac{1}{4}} \text{ \&c.}$$

Ex. 3.

Suppoſe $y^3 + aay + axy - x^3 - 2a^3 = 0$, *to find* y.

Put Ax_n for y, and the equation becomes $A^3x^{3n} + aaAx^n + aAx^{n+1} - x^3 - 2a^3x^0 = 0$.

For an aſcending ſeries.

Put the leaſt indices $n = 0$, and the indices become 0, 0, 1, 3, 0; and the differences 1, 3; and r, s, t, &c. = 1, 2, 3, 4, 5, *&c.* and the ſeries $y = A + Bx + Cx^2 + Dx^3$ &c. Then

$$\begin{array}{rclllll}
y^3 & = & A^3 & + 3A^2Bx & + 3AB^2x^2 & + 3A^2Dx^3 & \text{\&c.} \\
 & & & & + 3A^2C & + B^3 & \\
 & & & & & + 6ABC & \\
+ a^2y & = & a^2A & + aaBx & + aaCx^2 & + aaDx^3 & \\
+ axy & = & & + aAx & + aBx^2 & + aCx^3 & \\
- x^3 & = & & & & - x^3 & \\
- 2a^3 & = & - 2a^3 & & & &
\end{array}$$

Then equating the coefficients, $A^3 + aaA - 2a^3 = 0$, and extracting the root, $A = a$; alſo $3A^2B + aaB + aA = 0$, and $B = -\frac{1}{4}$. In like manner $C = \frac{1}{64a}$ and $D = \frac{131}{512aa}$, &c. Whence $y = a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^3}{512aa}$ &c.

Ex. 4.

Let $y^3+y^2+y-x^3=0$, *to find* y *in a deſcendng ſeries.*

Putting Ax^n for y, the equation becomes $A^3x^{3n}+A^2y^{2n}+Ay^n-x^3=0$. Put $3n=3$, or the greateſt indices. Then $n=1$, and all the ndexes are 3, 2, 1, 3; and the differences -1, -2; and the ſeries -1, -2, -3, -4, *&c.* and $=Ax+B+Cx^{-1}+Dx^{-2}$ &c. Then

$$\begin{array}{l|lllll}
y^3 & =A^3x^3 & +3A^2Bx^2 & +3A^2Cx & +3A^2D & \\
 & & & +3ABB & +6ABC & \&c. \\
 & & & & +B^3 & \\
+y^2 & & +A^2x^2 & +2ABx & +BB & \&c. \\
 & & & & +2AC & \\
+y & \ldots\ldots & \ldots\ldots & +Ax\ldots & +B & \&c. \\
-x^3 & -x^3 & & & &
\end{array}$$

Then equating the coefficients, $A^3=1$, and $A=1$. Likewiſe $B=-\frac{1}{3}$, $C=-\frac{2}{9}$, $D=\frac{7}{81}$, *&c.* and therefore $y=x-\frac{1}{3}-\frac{2}{9x}+\frac{7}{81xx}$ &c.

2 RULE.

Aſſume $y=Ax^n+Bx^{n+r}+Cx^{n+2r}+Dx^{n+3r}$ &c. and having found n, and put its value into the indices, as in Rule 1; ſet them down in order, and ſubtract each of them from the next greater; and you will have a ſeries of differences. Then find the greateſt number, which will meaſure all theſe differences; and this is the value of r, which muſt be affirmative in an aſcending ſeries, or when x is ſmall; and negative, in a deſcending one, when x is great. Then the values of n and r muſt be ſubſtituted in the aſſumed ſeries.

The proceſs muſt then go on as in Rule 1; and if there be any ſuperfluous terms, which will be known by ſome of the coefficients A, B, C coming out $= 0$; theſe terms muſt be thrown out of the ſeries, and the operation begun anew.

Ex. 5.

Let $y^3 - axy + x^3 = 0$, *be given.*

Put Ax^n for y, and the equation becomes $A^3x^{3n} - aAx^{n+1} + x^3 = 0$. Let $n + 1 = 3$, and $n = 2$; and the indices are 6, 3, 3; that is, 3, 6. Then $6 - 3 = 3$, then $r = 3$; and the leaſt indices being compared, the ſeries will be an aſcending one, which is this $y = Ax^2 + Bx^5 + Cx^8 + Dx^{11}$ &c. which ſubſtituted in the given equation will be as follows:

$$\begin{array}{r|lllll} y^3 & & + A^3x^6 & + 3A^2Bx^9 & \begin{array}{l} + 3A^2Cx^{12} \\ + 3ABB \end{array} & \&c. \\ - axy & - aAx^3 & - aBx^6 & - aCx^9 & - aDx^{12} & \\ + x^3 & + 1x^3 & & & & \end{array}$$

Then $aA = 1$, and $A = \frac{1}{a}$, $B = \frac{1}{a^4}$, $C = \frac{3}{a^7}$, $D = \frac{12}{a^{10}}$, &c. Whence $y = \frac{x^2}{a} + \frac{x^5}{a^4} + \frac{3x^8}{a^7} + \frac{12x^{11}}{a^{10}}$ &c.

Ex. 6.

Let $y^5 - by^2 + 9bx^2 - x^3 = 0$.

Subſtitute Ax^n for y and the equation is $A^5x^{5n} - bA^2x^{2n} + 9bx^2 - x^3 = 0$. Put $2n = 2$;

whence $n = 1$, and the indices are 5, 2, 2, 3; and the differences 1, 2. Whence $r = 1$. Therefore $y = Ax + Bx^2 + Cx^3 + Dx^4$ &c. Then

$$\begin{array}{l|llll}
y^5 & & & & +A^5x^5 \\
-by^2 & -bA^2x^2 & -2bABx^3 & -2bACx^4 & -2bADx^5 \\
 & & & -bBB & -2bBC \\
+9bx^2 & +9bx^2 & & & \\
-x^3 & & -x^3 & &
\end{array}$$

Here $bA^2 = 9b$; and $A = 3$: alſo $B = -\frac{1}{6b}$. $C = -\frac{1}{216bb}$, $D = \frac{81}{2b} - \frac{1}{3888b^3}$. Whence

$$y = 3x - \frac{x^2}{6b} - \frac{x^3}{216bb} + \overline{\frac{81}{2b} - \frac{1}{3888b^3}} \times x^4$$

&c.

3 RULE.

If the equation determining A, be an adfected equation, which has ſeveral equal roots or values of A, then you muſt divide the leaſt remainder, found by Rule 1, by the number of equal roots, one of which you take for A; and take this quotient for another remainder. Or elſe divide r found by Rule 2, by that number, and make uſe of the quotient, inſtead of r.

Ex. 7.

Let $y^9 - xy^3 + 2x^2y^2 - x^3y - x^{14} = 0$, *to find* y.

Put Ax^n for y, and the equation becomes $A^9x^9 - A^3x^{3n+1} + 2A^2x^{2n+2} - Ax^{n+3} - x^{14} = 0$. Let $3n + 1 = 2n + 2$; whence $n = 1$, then the indices are 9, 4, 4, 4, 14. But the ſum of the coefficients for the leaſt index 4, is $-A^3 + 2A^2 - A = 0$, or $A^2 - 2A + 1 = 0$, which equation has two equal roots $A = 1$, and $A = 1$. Now the difference of the indexes will be 5, 10; therefore divide 5 by 2, gives $\frac{5}{2}$, and we have $\frac{5}{2}$, 5, 10 for the differences. Therefore r, s, t, &c. will be $\frac{5}{2}$, 5, $7\frac{1}{2}$, 10, *&c.* Or (Rule 2) $r = 5$; therefore $\frac{r}{2} = \frac{5}{2}$ to be taken for r; whence the ſeries will be

$$y = Ax + Bx^{3\frac{1}{2}} + Cx^{6} + Dx^{8\frac{1}{2}} \text{ \&c. Then}$$

$$\begin{array}{l|llll}
y^{9} & & & + A^{9}x^{9} & \\
- xy^{3} & - A^{3}x^{4} & - 3A^{2}Bx^{6\frac{1}{2}} & - 3A^{2}Cx^{9} & \\
 & & & - 3AB^{2} & \\
+ 2x^{2}y^{2} & + 2A^{2}x^{4} & + 4ABx^{6\frac{1}{2}} & + 4ACx^{9} & \\
 & & & + 2BB & \\
- x^{3}y & - Ax^{4} & - Bx^{6\frac{1}{2}} & - Cx^{9} & \\
- x^{14} & & & & - \text{\&c.}
\end{array}$$

Hence $-A^{3} + 2A^{2} - A = 0$, and $A = 1$; $4B - 4B = 0$, and B may be taken at pleaſure. Suppoſe $B = -1$, then $1 - 3C - 3 + 4C + 2 - C = 0$, or $4C = 4C$, and C may be taken at pleaſure. Let $C = 1$; then $y = x - x^{3\frac{1}{2}} - x^{6}$ &c.

Or thus; In the ſecond equation, $4B = 4B$; which concludes nothing; alſo $1 - 3C - 3BB + 4C + 2BB - C = 0$; that is, $1 - BB = 0$, and $B = 1$ or -1, *&c.*

Ex. 8.

Let $A^{4}y^{2} - 2a^{4}xy + a^{4}x^{2} + x^{4}y^{2} = 0$.

Put Ax^{n} for y, and the indices become $2n$, $n + 1$, 2, $2n + 4$. Let $2n = 2$, or $n = 1$, and the indices are 2, 2, 2, 6; and the difference 4. The equation is $a^4A^2x^{2n} - 2a^4Ax^{n+1} + a^4x^2 - A^2x^{2n+4}$ or $a^4A^2x^2 - 2a^4Ax^2 + a^4x^2 - A^2x^6 = 0$, where the coefficient of the firſt term is $A^2 - 2A + 1 = 0$, which has two equal roots $A = 1$! Therefore divide the difference 4 by 2, and the quotient 2 is r or the common difference; whence the ſeries is $y = Ax + Bx^3 + Cx^5 + Dx^7$ &c. $= 0$. Then

$$\begin{array}{l|llll}
a^4yy & a^4A^2x^2 & +2a^4ABx^4 & +2a^4ACx^6 & +2a^4ADx^8 \\
 & & & +a^4BB & +2a^4BC \\
 & & & & \&c. \\
\hline
-2a^4xy & -2a^4Ax^2 & -2a^4Bx^4 & -2a^4Cx^6 & -2a^4Dx^8 \\
\hline
+a^4xx & +a^4x^2 & & & \\
-x^4y^2 & & & -A^2x^6 & -2ABx^8
\end{array}$$

Hence $A^2 - 2A + 1 = 0$, and $A = 1$; again, $B = B$, and B may be taken at pleaſure. Suppoſe $B = \frac{1}{aa}$. Again, $0C + a^4B^2 - 1 = 0$, or $0C = 1 - 1 = 0$, and C may be taken at pleaſure. Let $C = \frac{1}{a^4}$. Then $0D = 2AB - 2a^4BC = 0$, and D may be taken at pleaſure. Let $D = \frac{1}{a^6}$, &c.

Then $y = x + \frac{x^3}{aa} + \frac{x^5}{a^4} + \frac{x^7}{a^6} +$ &c.

Or rather thus, when A is determined to be 1, the firſt and third lines vaniſh; whence $a^4BB = A^2 = 1$, and $B = \frac{1}{aa}$; alſo $2a^4BC = 2AB$, and $C = \frac{1}{a^4}$, &c.

4 RULE.

If the quantity forming the ſeries (x) be nearly equal to ſome given quantity, put a new letter + that quantity for it, and ſubſtitute it in the equation;

tion; then find the root in an aſcending ſeries of the new letter. Or if the quantity (x) be very great, and the ſeries for y is to aſcend by x'es. Take ſome quantity nearly equal to x, and ſubſtitute the ſum of that and a new letter for x.

Ex. 9.

Let $y^3 + aay - x^3 = 0$, *where* $x = \frac{2}{3}a$, *nearly.*

Put $\frac{2}{3}a - v = x$. Then $x^3 = \frac{8}{27}a^3 - \frac{4}{3}aav + 2av^2 - v^3$, then $y^3 + aay - \frac{8}{27}a^3 + \frac{4}{3}aav - 2av^2 + v^3 = 0$. Let $y = Av^n$, then the indices are $3n$, n, 0, 1, 2, 3. Let $n = 0$, then the indices become 0, 0, 0, 1, 2, 3; and $y = A + Bv + Cv^2 + Dv^3$ &c. Then

$$\begin{array}{l|llll}
y^3 & A^3 & +\ 3A^2Bv & +\ 3A^2Cv^2\ \&c. & \\
 & & & +\ 3ABB & \\
+\ aay & aaA & +\ aaBv & +\ aaCv^2 & \\
-\ \frac{8}{27}a^3 & -\ \frac{8}{27}a^3 & & & \\
+\ \frac{4}{3}a^2v & & +\ \frac{4}{3}a^2v & & \\
-\ 2av^2 & & & -\ 2av^2 & \\
+\ v^3 & & & & +\ \&c.
\end{array}$$

Then $A^3 + 2\frac{1}{3}aaA - \frac{8}{27}a^3 = 0$, let $A = r$. Alſo $B = -\frac{4}{3}$; and $3A^2C + 3ABB + aaC = 2a$. Whence $C = \frac{2a - 5\frac{1}{3}r}{3rr + aa}$, &c.

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Ex. 10.

Let $y^4 - x^2y^2 + xy^2 + 2y^2 - 2y + 1 = 0$, *where* $x = 2$ *ery near.*

Let $x = 2 + z$, which ſubſtituted for x, there rifes $y^4 - z^2y^2 - 3zy^2 - 2y + 1 = 0$.

Let $y = Az^n$, then the equation is $A^4z^{4^n}$ $-A^2z^{2n+2} - 3A^2z^{2n+1} - 2Az^n + 1 = 0$. Let $= 0$, and the indices are 0, 2, 1, 0, 0; and he differences 1, 2, 3, 4, *&c.* whence $y = a +$ $z + Cz^2 + Dz^3$ &c. Then

$$\begin{array}{l|llll}
y^4 & A^4 & +4A^3Bz & +4A^3Cz^2 & \&c. \\
 & & & +6A^2BB & \\
-z^2y^2 & & & -\ AAz^2 & \\
-3zy^2 & & -3A^2z & -6ABz^2 & \\
-2y & -2A & -2Bz & -2Cz^2 & \\
+1 & +1 & & &
\end{array}$$

Here $A^4 - 2A + 1 = 0$, and $A = 1$; alſo $B - 2B = 3$, and $B = \frac{3}{2}$; and $4C + 6BB - 1 -$ $B - 2C = 0$, and $C = -\frac{7}{4}$, *&c.* and $y = 1 +$ $\frac{3}{2}z - \frac{7}{4}zz$ &c.

Cor. 1. *In all theſe caſes of extracting roots, the eries muſt be made to converge, or elſe they are of no ſe. For in a converging ſeries, the terms grow continually leſs and leſs, and ſo approach nearer and earer to the true root, till the difference is as ſmall s you will. But a diverging ſeries always runs farther from the root, and therefore gives a falſe value hereof.*

Cor. 2. *If y be denoted by a ſeries of x aſcending; the leſſer x is, the faſter the ſeries converges.*

And in a ſeries of x deſcending; the greater x is, the faſter likewiſe it converges. Therefore we are ſo to contrive the ſeries, that we may have the leaſt quantity in the numerators, or the greateſt in the denominators.

Cor. 3. *If the equation for finding the firſt term* A, *be an adfected equation; as many roots or different values of* A, *as that equation has, ſo many different ſeries will ariſe. For the firſt term* A *being different in each, the coefficients* B, C, D, *depending thereon, will alſo be different. Likewiſe, if two roots are equal, the ſecond term will vaniſh, and the coefficient* B *will be found in the third, which will be a quadratic equation. And if there be three equal values of* A, *the ſecond and third equations vaniſh, and the fourth contains a cubic equation of* B, *&c.*

Cor 4. *An equation will alſo admit of ſeveral different ſeries for the roots, according to the different values aſſumed for* n. *Alſo there are other equations that are impoſſible, and will admit of no roots.*

Cor. 5. *When the firſt equation, or that for determining* A, *has ſeveral equal roots; then the values of* r, s, t, &c. *muſt be divided by that number. Or, which is the ſame thing, the indices of* x (r, s, t,) *found by Rule* 1, *muſt have others interpoſed betwee them, according to the number of equal roots. As fo two equal roots, the ſeries* $Ax^n + Bx^{n+r} + Cx^{n+}$ &c. *muſt be reduced to this,* $Ax^n + Bx^{n+\frac{1}{2}r} + Cx^{n+r} + Dx^{n+\frac{1}{2}s} + Ex^{n+s}$ &c. *If this be no done, the ſecond term* B *will be infinite, and all th following ones.*

Cor. 6. *If the ſeries* $A + \overline{B+C} \times z + \overline{D+E+} \times z^2 + \overline{G+H+I+K} \times z^3$ &c. $= 0$, z *bein an indetermined quantity; then whatever value is p upon* z, *it will be* $A = 0$, $B + C = 0$, $D + E + F =$ $G + H + I + K = 0$, *&c.*

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For this being a general equation, where z may be of any value; therefore put $z = 0$, and then will A $= 0$, and $\overline{B + C} \times z + \overline{D + E + F} \times z^2$ &c. $= 0$, divide by z, then $B + C + \overline{D + E + F} \times z$ &c. $= 0$. Again, put $z = 0$, and then $B + C = 0$; whence $\overline{D + E + F} \times z + \overline{G + H + I + K} \times z^2$ &c. $= 0$. Divide again by z, and $D + E + F + \overline{G + H + I + K} \times z = 0$. Again, put $z = 0$, then $D + E + F = 0$, nd $\overline{G + H + I + K} \times z = 0$, and $G + H + I + K = 0$, &c.

Reverſion of ſeries, and the extracting the roots f all infinite ſeries, depends upon this. For the oefficients of the ſeveral powers of the indetermined quantity, muſt be put $= 0$, or elſe the whole quation cannot vaniſh, as it ought to do. And his being done, the ſeveral aſſumed coefficients A, B, C are determined as in the problems above.

SCHOLIUM.

To find y in the ſeries $ay^{\mu} + by^{\mu+\nu} + cy^{\mu+2\nu}$ &c. $= fx^{\pi} + gx^{\pi+\varrho} + hx^{\pi+2\varrho}$ &c. Aſſume $= Ax^{n} + Bx^{n+r} + Cx^{n+s}$ &c. Then by ſubſtitution we get, $aA^{\mu}x^{\mu n} + bA^{\mu+n}x^{\mu n+\nu n}$ &c. $= fx^{\pi} + gx^{\pi+\varrho}$ &c. Whence making the leaſt ndices equal, $\mu n = \pi$; then $n = \frac{\pi}{\mu}$, and the differences will be $\frac{\pi\nu}{\mu}$, ϱ; &c. Then find q the greateſt common diviſor of $\frac{\pi\nu}{\mu}$ and ϱ; and the form of the ſeries will be

$$y = Ax^{\frac{\pi}{\mu}} + Bx^{\frac{\pi}{\mu}+q} + Cx^{\frac{\pi}{\mu}+2q} + Dx^{\frac{\pi}{\mu}+3q}$$

&c. in which the coefficients will be determined as before.

SECT. VII.

Some general and fundamental Problems, uſeful and neceſſary in algebraical calculations.

PROBLEM LXV.

The ſum and difference of two quantities being given; to find the quantities.

LET $s =$ the ſum
$d =$ the difference
$a =$ greater quantity
$e =$ the leſſer.

then $a + e = s$, by the problem.
and $a - e = d$.

then $2a = s + d$ by addition
and $2e = s - d$ by ſubtracting.

Whence $a = \frac{s + d}{2}$, and $e = \frac{s - d}{2}$

or $a = \frac{1}{2}s + \frac{1}{2}d$, and $e = \frac{1}{2}s - \frac{1}{2}d$.

Cor. 1. *Half the ſum added to half the differenc of two quantities, is equal to the greater.*

Cor. 2. *Half the difference of two quantities, taken from half the ſum, gives the leſſer quantity.*

PROBLEM LXVI.

To find out the leaſ common dividend, or the leaſ quantity, that can be divided by ſeveral give quantities.

RULE.

Reſolve each of the quantities into all the ſimple diviſors contained therein, by firſt dividing b
th

the leaſt, and then by the next, and ſo on, till they are all exhauſted; and collect theſe diviſors together for each quantity. Then if there be any diviſors in the ſecond quantity which is not in the firſt, multiply the firſt by ſuch diviſors. Likewiſe, if there be any diviſors in the third quantity which is not in this laſt, multiply it thereby, or put them into that quantity. Likewiſe ſuch diviſors as are in the fourth quantity and are not in this laſt, muſt be put into it, and ſo on. And laſtly, all theſe diviſors, in this laſt quantity muſt be multiplied together for the leaſt common dividend.

Or ſhorter thus,

Divide the product of any two of the quantities by their greateſt common diviſor, (found by Prob. x. Sect. II.) take this quotient and a third quantity, and divide their product, by their greateſt common diviſor. Take this quotient and another quantity, and proceed as before; and ſo on to the laſt quantity. And the laſt quotient will be the leaſt common dividend.

Ex. 1.

What is the greateſt common dividend of a^2bc, *and* $2ab^2d$.

The diviſors of a^2bc are a, a, b, c
of $2abbd$ are 2, a, b, b, d.

Here 2, b, d are in the laſt but not in the firſt; therefore $a \times a \times b \times c \times 2bd$, or $2aabbcd$ is the leaſt common dividend.

Or thus,

The greateſt common meaſure is ab, then the product is $2a^3b^3cd$. $ab)2a^3b^3cd(2aabbcd$ the leaſt dividend.

Ex. 2.

Let $ab + cd$ *and* $ac + bd$ *be propoſed.*

Theſe have no diviſors but 1. Therefore $\overline{ab + cd} \times \overline{ac + bd}$ or $aabc + abbd + accd + bcdd$, is the dividend required.

Ex. 3.

Let $3a^2b$, $a^3 + a^2b$, and $aa - bb$, *be given.*

The greateſt common diviſor of $3a^2b$ and $a^3 + a^2b$ is aa. Then

$$aa)\,3a^5b + 3a^4bb\,(3a^3b + 3aabb.$$

Then the greateſt common diviſor of $3a^3b + 3aabb$ and $aa - bb$, is $a + b$; then $\overline{3a^3b + 3aabb} \times \overline{aa - bb}$ divided by $a + b$ is

$$a + b)\,3a^5b + 3a^4bb - 3a^3b^3 - 3a^2b^4\,(3a^4b - 3a^2b^3$$

the leaſt common dividend.

Ex. 4.

Let the given quantities be $a^4 - b^4$, $aa + ab$, $a^4 + a^2b^2$, *and* $a + b$.

Theſe quantities reſolved into their diviſors are $\overline{aa+bb} \times \overline{a+b} \times \overline{a-b}$, $a \times \overline{a+b}$, $a \times a \times \overline{aa+bb}$, and $a + b$. Now becauſe there is one factor a in the ſecond which is not in the firſt, put it in the firſt, which becomes $\overline{aa + bb} \times \overline{a + b} \times \overline{a - b} \times a$, the leaſt dividend for the firſt two quantities.

Likewiſe, there is a, one factor in the third, which is not in this laſt; let it be inſerted, and it becomes $\overline{aa + bb} \times \overline{a + b} \times \overline{a - b} \times aa$, the leaſt dividend for three quantities.

Laſtly, Since $a + b$ the laſt given quantity is in the laſt dividend; it will be the dividend for all four; that is, $\overline{aa + bb} \times \overline{a + b} \times \overline{a - b} \times aa$, or

$a^6 - a^4bb$ is the leaſt common dividend for the four given quantities.

SCHOLIUM.

All the ſimple diviſors of a quantity, are found the ſame way, as in Prob. 6, 7. Chap. iv. B. II. Arithmetic.

PROBLEM LXVII.

The ſum and difference of two quantities being given; to find the difference of their ſquares.

Let $s =$ ſum, $d =$ difference, A $=$ greater quantity, E $=$ the leſſer. Then $A = \frac{s+d}{2}$, and $E = \frac{s-d}{2}$ (Prob. lxv). Whence

$$AA = \frac{ss + 2sd + dd}{4}$$

$$\text{and } EE = \frac{ss - 2sd + dd}{4}$$

$$\text{and } AA - EE = \frac{4sd}{4} = sd.$$

Cor. *The product of the ſum and difference of two quantities, is equal to the difference of their ſquares.*

PROBLEM LXVIII.

Two quantities being given to find the ſquare of the ſum.

Let a be the greater quantity, e the leſſer; then the ſum is $a + e$; and $a + e$ being ſquared is $aa + 2ae + ee$.

Cor. 1. *Hence the ſquare of the ſum of two quantities is equal to the ſum of the ſquares of the quantities, increaſed by double their product.*

Cor. 2. *The ſquare of the ſum of any number of quantities,* $a + b + c$ *&c. is equal to the ſum of all the ſquares, together with twice the ſum of all the products of every two.*

For by this Prob. $\overline{a+b+c}|^2 = \overline{a+b}|^2 + 2\times\overline{a+b}\times c+cc$; that is $aa+2ab+bb+2ac+2bc+cc$, and ſo for more quantities.

Schol. By the ſame way, theorems may be found for the cube of the ſum of two or more quantities.

PROBLEM LXIX.

Two quantities being given to find the ſquare of their difference.

Let a be the greater, e the leſſer; then the difference is $a - e$, which being ſquared, produces $aa - 2ae + ee$.

Cor. *Hence the ſquare of the difference of two quantities, is equal to the ſum of their ſquares abating twice their product.*

Schol. By the ſame method a rule may be found for the cube of the difference of two quantities.

PROBLEM LXX:

The ſum and difference of two quantities being given; to find their rectangle.

Let $s =$ ſum, $d =$ difference, A the greater, E the leſſer. Then $A + E = s$, and $A - E = d$; and adding theſe equations $2A = s + d$; and ſubtracting, $2E = s - d$. Then $2A \times 2E$ or $4AE = \overline{s+d} \times \overline{s-d} = ss - dd$, and $AE = \dfrac{ss-dd}{4}$.

Cor. *The ſquare of the ſum, leſs the ſquare of the difference of two quantities, is equal to four times their rectangle.*

PROBLEM LXXI.

Given the n^{th} power of the binomial $a + b$; to find the difference between the ſquare of the ſum of the odd terms, and the ſquare of the ſum of the even terms.

The n^{th} power of $a + b$, that is $\overline{a+b}|^{n} = a^{n} + na^{n-1}b + n.\frac{n-1}{2}a^{n-2}bb + n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^{3}$ &c. Put A, B, C, D, E, &c. for the firſt, ſecond, third, fourth, &c. terms. Then $A + C + E$ &c. = ſum of the odd terms; and $B + D + F$ &c. = ſum of the even terms. But $\overline{A + C + E\ \&c.}|^{2} - \overline{B + D + F}|^{2} = \overline{A + B + C + D + E\ \&c.} \times \overline{A - B + C - D + E\ \&c.} =$

$$\overline{a^{n} + na^{n-1}b + n.\frac{n-1}{2}a^{n-2}bb + \&c.} \times$$

$$\overline{a^{n} - na^{n-1}b + n.\frac{n-1}{2}a^{n-2}bb - \&c.}$$

$$= \overline{a+b}|^{n} \times \overline{a-b}|^{n} = \overline{aa-bb}|^{n}.$$

Cor. 1. *Hence* $\overline{aa-bb}|^{n} =$

$$\overline{a^{n} + \frac{n.n-1}{2}a^{n-2}bb + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.a^{n-4}b^{4}\ \&c.}\Big|^{2}$$

$$\overline{- na^{n-1}b + n.\frac{n-1}{2}.\frac{n-2}{3}.a^{n-3}b^{3} + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{n-4}{5}a^{n-5}b^{5}\ \&c.}\Big|^{2}$$

Cor. 2. $\overline{aa-bb}|^{2} = \overline{aa+bb}|^{2} - \overline{2ab}|^{2}$

Cor. 3. $\overline{aa-bb}|^{3} = \overline{a^{3} + 3abb}|^{2} - \overline{3aab + b^{3}}|^{2}$.

Cor. 4. $\overline{aa-bb}|^{4} = \overline{a^{4}+6aabb+b^{4}}|^{2} - \overline{4a^{3}b+4ab^{3}}|^{2}$.

Cor. 5. $\overline{aa-bb}|^{5}$

$= \overline{a^{5}+10a^{3}bb+5ab^{4}}|^{2} - \overline{5a^{4}b+10a^{2}b^{3}+b^{5}}|^{2}$. &c.

PROBLEM LXXII.

To find the n^{th} *root of the binomial ſurd* A ± B, *where either* A *or* B *is a ſurd ſquare root, the other rational.*

Suppoſe $x + v = \sqrt[n]{A + B}$, then by involution $A + B = x + nx^{n-1}v + n.\frac{n-1}{2}x^{n-2}vv + n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}v^{3}$ &c. Suppoſe v a ſurd ſquare root, and put the odd terms of the ſeries = A, and the even ones = B; that is $x^{n} + n.\frac{n-1}{2}x^{n-2}vv$ &c. = A, and $nx^{n-2}v + n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}v^{3}$ &c. = B. Then $\overline{x^{n} + n.\frac{n-1}{2}x^{n-2}vv \;\&c.}|^{2}$ $\overline{- nx^{n-1}v + n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}v^{3} \;\&c.}|^{2} = A^{2} - B^{2} = D$ by ſubſtitution. That is (by Cor. 1. Prob. lxxi.), $\overline{xx-vv}|^{n} = D$, and $xx - vv = D^{\frac{1}{n}}$. Therefore $vv = xx - D^{\frac{1}{n}}$. Whence this equation,

$$x^{n} + n.\frac{n-1}{2}x^{n-2} \times xx - D^{\frac{1}{n}} + \frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}x^{n-4} \times \overline{xx - D^{\frac{1}{n}}}\Big|^{2} + \&c. = A.$$

Which affected equation, by a few trials, will give x, and then v will be had by the equation, $v = \pm\sqrt{xx - \sqrt{D}}$; and $x \pm v = \sqrt[n]{A \pm B}$, as required.

Cor. *Hence if* $x \pm \sqrt{y} = \sqrt[n]{A \pm B}$, *and* $A^2 - B^2 = D$; then $x^n + n \cdot \frac{n-1}{2} \times \frac{xx - D^{\frac{1}{n}}}{xx} P + \frac{n-2}{3} \cdot \frac{n-3}{4} \times \frac{xx - D^{\frac{1}{n}}}{xx} Q + \frac{n-4}{5} \cdot \frac{n-5}{6} \times \frac{xx - D^{\frac{1}{n}}}{xx} R$ *&c.* $= A$; *and* $\sqrt{y} = \sqrt{xx - \sqrt[n]{D}}$. Where P, Q, R, are the foregoing terms.

Or thus,

Since $x^n + n \cdot \frac{n-1}{2} x^{n-2} vv$ &c. $= A$.

and $nx^{n-1} v + n. \frac{n-1}{2} : \frac{n-2}{3} x^{n-3} v^3$ &c. $= B$.

by adding,

$$x^n + nx^{n-1} v + n \cdot \frac{n-1}{2} x^{n-2} v^2 \text{ \&c.} = A + B.$$

by subtracting,

$$x^n - nx^{n-1} v + n \cdot \frac{n-1}{2} x^{n-2} v^2 - \text{\&c.} = A - B.$$

Therefore by extraction,

$$x + v = \sqrt[n]{A + B}.$$

and $x - v = \sqrt[n]{A - B}$.

and $2x = \sqrt[n]{A + B} + \sqrt[n]{A - B}$.

$$2v = \sqrt[n]{A + B} - \sqrt[n]{A - B}.$$

whence $x = \frac{\sqrt[2]{A+B} + \sqrt[2]{A-B}}{2}$,

$$v = \frac{\sqrt[2]{A+B} - \sqrt[2]{A-B}}{2}.$$

Therefore $x \pm v$ will be had, at leaſt in decimals.

Cor. Hence, $\sqrt[2]{A \pm B} = \frac{\sqrt[2]{A+B} + \sqrt[2]{A-B}}{2}$

$\pm \frac{\sqrt[2]{A+B} - \sqrt[2]{A-B}}{2}$.

Ex. 1.

Extract the ſquare root of $11 + 6\sqrt{2}$.

Here $A = 11$, $B = 6\sqrt{2}$, $AA - BB = 49 = D$, and $\sqrt{D} = 7$. Therefore $x^2 + \frac{xx-7}{xx}x^2 = 11$, or $2xx - 7 = 11$, and $xx = 9$, or $x = 3$. Likewiſe $\sqrt{v} = \pm\sqrt{9-7} = \pm\sqrt{2}$, and $x + \sqrt{v} = 3 + \sqrt{2}$ the root.

Or thus.

$A + B = 11 + 6\sqrt{2} = 19.484$, and $A - B = 2.516$. Whence $x = \frac{\sqrt{19.484} + \sqrt{2.516}}{2} = \frac{4.414 + 1.586}{2} = 3$, and $v = \frac{4.414 - 1.586}{2} = 1.414 = \sqrt{2}$, and $x + v = 3 + \sqrt{2}$, the root.

Ex. 2.

Suppoſe $37 - 20\sqrt{3}$ *be given, to extract the ſquare root.*

Here $A = 37$, $B = 20\sqrt{3}$, $AA - BB = 169 = D$, and $\sqrt{D} = \pm 13$. Therefore $2xx - 13 = 37$, and $xx = 25$, or $x = 5$.

Alſo $\sqrt{y} = -\sqrt{25 - 13} = -\sqrt{12}$, and $x + \sqrt{y} = 5 - \sqrt{12}$. Or the root may be $\sqrt{12} - 5$, putting $\sqrt{D} = -13$.

Or thus,

$$x = \frac{\sqrt{71.64} + \sqrt{2.36}}{2} = \frac{10.00}{2} = 5.$$

$$v = \frac{\sqrt{2.36} - \sqrt{71.64}}{2} = -\frac{6.92}{2} = -3.46$$

$= -2\sqrt{3}$, and $x + v = 5 - 2\sqrt{3}$ the root.

Ex. 3.

Let $7 - 5\sqrt{2}$ *be given to find the cube root.*

Here $A=7$, $B=5\sqrt{2}$, $A^2-B^2=-1=D$, and $D^{\frac{1}{3}} = -1$.

Then $x^3 + 3 \times \frac{xx+1}{xx} x^3 = 7$, or $4x^3 + 3x = 7$, and the root of this equation is $x = 1$.

Alſo $\sqrt{y} = -\sqrt{xx+1} = -\sqrt{2}$, and $x + \sqrt{y} = 1 - \sqrt{2}$.

Or thus,

$\sqrt[3]{A+B} = \sqrt[3]{14.07} = 2.414$, and
$\sqrt[3]{A-B} = \sqrt[3]{-.07} = -.414$;

Then $x = \frac{-.414 + 2.414}{2} = 1$.

And $v = \frac{-.414 - 2.414}{2} = -1.414 = -\sqrt{2}$ for here B is negative; therefore $x + v = 1 - \sqrt{2}$.

Ex. 4.

What is the cube root of $25 + \sqrt{968}$.

Here $A = 25$, $B = \sqrt{968}$, $AA - BB = -343 = D$, and $\sqrt[3]{D} = -7$.

Then $x^3 + 3x^3 + 21x = 25$, and $x = 1$.

And $\sqrt{y} = \sqrt{8}$, and the root, $1 + \sqrt{8}$.

Ex.

Ex. 5.

Extract the cube root of $-10 + \sqrt{-243}$.

Here $A^2 - B^2 = 100 + 243 = 343 = D$, and $D^{\frac{1}{3}} = 7$. Therefore $x^3 + 3x \times xx - 7 = -10$, or $4x^3 - 21x = -10$, and the root is $x = 2$, whence v or $\sqrt{y} = \sqrt{4 - 7} = \sqrt{-3}$, and $x + \sqrt{y} = 2 + \sqrt{-3}$, as required.

In like manner the cube root of $-10 - \sqrt{-243}$ is $2 - \sqrt{-3}$.

Ex. 6.

Extract the 5th root of $843 - 589\sqrt{2}$.

Here $AA - BB = 16807$, and $D^{\frac{1}{5}} = 7$.

And $16x^5 - 140x^3 + 245x = 843$, and the root is $x = 3$; and $\sqrt{y} = -\sqrt{9 - 7} = -\sqrt{2}$, and $x + \sqrt{y} = 3 - \sqrt{2}$ the root required.

Ex. 7.

What is the 7th root of $568 + 328\sqrt{3}$.

Here $A^2 - B^2 = -128 = D$, and $\sqrt[7]{D} = -2$. Then $A + B = 1136.112$, $A - B = -.112$, and

$$x = \frac{\sqrt[7]{1136.112} - \sqrt[7]{-.112}}{2} = \frac{2.732 - .732}{2} = 1,$$

And

$$v = \frac{\sqrt[7]{1136.112} + \sqrt[7]{-.112}}{2} = \frac{2.732 + .732}{2} = 1.732 = \sqrt{3},$$

and $x + v = 1 + \sqrt{3}$ = the root.

SCHOLIUM.

In the former method, if $\sqrt[n]{D}$ is not rational, neither member of the root will be rational, and in the ſecond, if neither the ſum nor difference of $\sqrt[n]{A+B}$ and $\sqrt[n]{A-B}$, is rational; neither member of the root will be ſo: and in theſe caſes the rules are of no uſe. Logarithms will be uſeful here in finding theſe roots, being exact enough in finding whether any of the quantities be rational or not. When none of theſe quantities are rational, multiply the given equation by ſome number, till $\sqrt[n]{D}$, or $\sqrt[n]{A+B} \pm \sqrt[n]{A-B}$, comes out rational; then extract the root as before. But remember to divide the values of x, v, at laſt, by the root of that number. Thus $22 + \sqrt{486}$ has not ſuch a cube root; but multiply by 2, and then $\frac{44 + \sqrt{1944}}{2}$, will have a cube root, for the numerator.

PROBLEM LXXIII.

To explain the ſeveral properties of (0) *nothing, and infinity.*

It is plain, nothing added to, or ſubtracted from, any quantity, makes it neither bigger nor leſs.

Likewiſe, if any quantity is multiplied by 0, that is, taken no times at all; the product will be nothing.

Let $\frac{b}{a} = q$; that is, let the quotient, of b divided by a, be q. Then if b remains the ſame, it is plain the leſs a is, the greater the quotient q will be. Let a be indefinitely ſmall beyond all bounds, then q will be indefinitely great beyond all bounds. Therefore when a is nothing, the quotient q will be infinite. Whence

Alſo ſince $\frac{b}{0} =$ infinity, therefore $b =$ nothing $\times$ infinity.

Let there be ſeveral geometrical proportionals, x, x^2, x^3, x^4, x^5, &c. If this ſeries be continued backwards, it will be x, 1, $\frac{1}{x}$, $\frac{1}{xx}$; that is, x^1, x^0, x^{-1}, x^{-2}, the indices continually decreaſing by 1. Then its plain x^0 is equal to 1, whatever x be; for it may ſtand univerſally for any thing. Therefore 0^0 is $= 1$.

Let x be an indefinitely ſmall quantity, beyond all conception; then in the ſeries x, x^2, x^3, &c. each term will be indefinitely greater than the following one. And when x is 0, then in the ſeries $\frac{1}{0}$, 0^0, 0^1, 0^2, *&c.* $\frac{1}{0}$ is infinite, and 0 is nothing, by what goes before. Therefore the mean 0^0 is a finite quantity. Suppoſe $= b$, whence $\frac{1}{0} \times 0 = bb$, that is $bb = \frac{1 \times 0}{0} = 1$, and $b = 1$, whence it is plain again, that (b) $0^0 = 1$.

Let $\frac{a}{1-1}$ or its equal $\frac{a}{-1+1}$ be an infinite quantity, then by actually dividing, $\frac{a}{1-1} = a + a + a + \frac{a}{1-1}$, and $\frac{a}{-1+1} = -a-a-a+\frac{a}{-1+1}$. Therefore $\frac{a}{1-1} + a + a + a$ &c. $= \frac{a}{1-1} - a - a - a$ &c. that is, an infinite quantity is neither increaſed nor decreaſed by finite quantities.

Cor. 1. *If* 0 *multiply any finite quantity, the product will be nothing.*

Cor. 2. *If* o *multiply an infinite quantity, the product is a finite quantity. Or a finite quantity is a mean proportional between nothing and infinity.*

For $0 \times \text{infinity} = b$.

Cor. 3. *If a finite quantity is divided by* o, *the quotient is infinite* ($\frac{b}{0} = \text{inf.}$).

Cor. 4. *If* o *be divided by* o, *the quotient is a finite quantity of ſome ſort.*

For (Co. 1.) $b \times 0 = 0$, and therefore $\frac{0}{0} = b$, a finite quantity, or nothing.

Cor. 5. *Hence alſo* $0^0 = 1$, *or the infinitely ſmall power, of an infinitely ſmall quantity, is infinitely near* 1.

Cor. 6. *Adding or ſubtracting any finite quantities to or from an infinite quantity, makes no alteration.*

Cor. 7. *Therefore in any equation, where are ſome quantities infinitely leſs than others; they may be thrown out of the equation.*

Cor. 8. *An infinite quantity may be conſidered either as affirmative or negative.*

For infinity $= \frac{b}{+0}$ or $\frac{b}{-0}$.

SCHOLIUM.

There is ſomething extremely ſubtle, and hard to conceive, in the doctrine of *infinites* and *nothings.* Yet although the objects themſelves are beyond our comprehenſion; yet we cannot reſiſt the force of demonſtration, concerning their powers, properties, and effects; which properties, under ſuch and ſuch conditions, I think, I have truly explained in this propoſition. Any metaphyſical notions, that go beyond theſe mathematical operations, are not the buſineſs of a mathematician. But thus much may be obſerved, that 0, in a mathematical ſenſe, never ſignifies abſolute nothing; but always nothing in relation to the object under conſideration. For illuſtration thereof, ſuppoſe we are conſidering the area contained between the baſe of a parallelogram and a line drawn parallel to the baſe. As this line draws nearer the baſe, the area diminiſhes; till at laſt, when the line coincides with the baſe, the area becomes nothing. So the area here degenerates into a line; which is nothing, or no part of the area. But it is a line ſtill, and may be compared with other lines.

PROBLEM LXXIV.

To find the value of a fraction, when the numerator and denominator, is each of them nothing.

1 RULE.

Conſider, from the nature of the queſtion propoſed, what quantities are infinitely greater than others, when they are all taken infinitely ſmall. Then throw out of the equation, all thoſe terms that are infinitely leſs than others; retaining only thoſe that are infinitely greater than the reſt; by which expunge one of the unknown quantities, and the value of the fraction will be known.

Ex. 1.

Let $x^3 + y^3 = axy$, *and* y *infinitely greater than* x, *when they vaniſh; to find the value of* $\frac{yy}{x}$, *when* x *and* y *are* $= 0$.

Here x^3 is infinitely leſs than axy or y^3, whence $y^3 = axy$, or $yy = ax$. Then $\frac{yy}{x} = \frac{ax}{x} = a$, the value of the fraction propoſed.

Ex.

Ex. 2.

If $2ax + xx = yy$, *what is the value of* $\frac{x}{yy}$, *when* x *and* $y = 0$, and y *infinitely greater than* x.

Here reject xx being infinitely leſs than the reſt; then $yy = 2ax$, and $\frac{x}{yy} = \frac{1}{2a}$.

Ex. 3.

What is the value of $\frac{y}{x}$, *when* $2ay + yy = rx$; y, x *being* $= 0$.

Here yy is infinitely leſs than $2ay$. Whence $2ay = rx$, and $\frac{y}{x} = \frac{r}{2a} = \frac{0}{0}$.

2 RULE.

Obſerve what the unknown quantity is equal to, when the numerator, *&c.* vaniſhes; put the unknown quantity = that value ± e, where e is ſuppoſed infinitely ſmall. Which being ſubſtituted for that unknown quantity, and the roots of all ſurds, extracted to a ſufficient number of places of e; at laſt you will have ſome terms in both the numerator and denominator, which will determine the value of the fraction.

Ex. 4.

What is the value of $\frac{a\sqrt{ax} - xx}{a - \sqrt{ax}}$, *when* $x = a$.

Put $x = a + e$, then expunging x; $\frac{a\sqrt{ax} - xx}{a - \sqrt{ax}}$ =

$$= \frac{a \times \overline{aa + ae}|^{\frac{1}{2}} - \overline{a + e}|^{2}}{a - \overline{aa + ae}|^{\frac{1}{2}}} = \frac{a \times : a + \frac{1}{2}e \text{ \&c.} - aa - 2ae \text{ \&c.}}{a - a - \frac{1}{2}e \text{ \&c.}} =$$

$$\frac{aa + \frac{1}{2}ae \text{ \&c.} - aa - 2ae \text{ \&c.}}{-\frac{1}{2}e} = \frac{-\frac{3}{2}ae}{-\frac{1}{2}e} = \frac{3ae}{e}$$

$= 3a$, the value of the fraction.

Ex. 5.

What is the value of $\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$, *when* $x = a$.

Let the fraction $= y$, and put $x = a - e$, then

$$y = \frac{\sqrt{2a^3 \times \overline{a - e} - \overline{a - e}|^4} - a\sqrt[3]{a^3 - aae}}{a - \sqrt[4]{a \times \overline{a - e}|^3}}.$$

But $\sqrt{2a^3 \times \overline{a - e} - \overline{a - e}|^4} = \sqrt{2a^4 - 2a^3e - a^4 + 4a^3e} = \overline{a^4 + 2a^3e}|^{\frac{1}{2}} = aa + ae$ &c. Alſo $a\sqrt[3]{a^3 - aae} = a \times \overline{a - \frac{1}{3}e} = aa - \frac{1}{3}ae$ &c. And $\sqrt[4]{a \times \overline{a - e}|^3} = \overline{a^4 - 3a^3e}|^{\frac{1}{4}} = a - \frac{3}{4}e$ &c. Whence

$$y = \frac{aa + ae \text{ \&c.} - aa + \frac{1}{3}ae \text{ \&c.}}{a - a + \frac{3}{4}e \text{ \&c.}} = \frac{\frac{4}{3}ae}{\frac{3}{4}e} = \frac{16a}{9}.$$

Ex. 6.

Let $\frac{a\sqrt[3]{4a^3 + 4x^3} - ax - aa}{\sqrt{2aa + 2ax} - x - a} = y$, *what is its value when* $x = a$.

Let $a - e = x$. And expunging x,

$$\frac{a\sqrt[3]{4a^3 + 4 \times \overline{a - e}^3} - aa + ae - aa}{\sqrt{2aa + 2 \times \overline{a - e}|^2} - x - a} = y.$$

But

$a\surd$

$a\sqrt[3]{4a^3+4\times\overline{a-e}|^3}=a\times\overline{8a^3-12aae+12aee}|^{\frac{1}{3}}=2aa-ae+\frac{1}{2}ee$ &c. And $\sqrt{2aa+2xx}=\overline{4aa-4ae+2ee}|^{\frac{1}{2}}$ $=2a-e+\frac{ee}{4a}$ &c. Whence

$$y=\frac{2aa-ae+\frac{1}{2}ee\ \&c.-2aa+ae}{2a-e+\frac{ee}{4a}\ \&c.-2a+e}=\frac{+\frac{1}{2}ee}{+\frac{ee}{4a}}=\frac{2aee}{ee}=2a.$$

Here, if I had gone no farther, than the firſt power of e, it is evident by inſpection, that all the terms would have vaniſhed; by which nothing could have been concluded.

SCHOLIUM.

If e remains at laſt in the numerator, the value of the fraction is 0, and if e remains in the denominator, the fraction is infinite. But if all the terms vaniſh out of both numerator and denominator, the ſeries muſt then be carried to more places, to have a ſolution.

PROBLEM LXXV.

To find two whole numbers x, y; in the equation $ax=by+c$, being in its leaſt terms: a, b, c, being given numbers.

RULE.

Let *wh.* ſtand for the words *a whole number.* Reduce the equation, then $x=\frac{by+c}{a}=$ *wh.* By an abridged fraction, I mean the fraction reſulting by throwing all whole numbers out of it, till the terms in the numerator be leſs than the deno-

minator.

minator. Thus let the fraction $\frac{by + c}{a}$ be abridged to $\frac{dy + f}{a}$. Then to find y.

The method conſiſts in leſſening the coefficient of y continually, till at laſt it becomes 1. And this is done by ſubtracting $\frac{dy + f}{a}$ or ſome multiple of it, from y, or any multiple of it, which comes very near it; that is, from $\frac{ay}{a}$, $\frac{2ay}{a}$, $\frac{3ay}{a}$, &c. or this from it. And the reſulting fraction abridged, or its neareſt multiple, is in like manner to be ſubtracted from the neareſt foregoing fraction; or from any *wh.* which is nearer; or this from that. And theſe *wh.* may be $\frac{ay}{a}$, $\frac{2ay}{a}$, &c. or $\frac{ay \pm a}{a}$, $\frac{ay \pm 2a}{a}$, $\frac{2ay \pm 3a}{a}$ &c. or any you can find, which has the neareſt coefficient to y. By this means the coefficient of y is continually leſſened, till at laſt we have $\frac{y + g}{a} = wh = p$; then will $y = ap - g$: where p may be any whole number taken at pleaſure. And y being known, x will be found from the given equation.

You muſt obſerve in this whole proceſs, to keep the ſame denominator a, throughout.

For whole numbers ſubtracted from one another, will always leave whole numbers. And whole numbers multiplied by whole numbers, will always produce whole numbers. And upon theſe principles the rule is founded.

Ex. 1.

Let $19x = 14y - 11$, *to find* x, y *in whole numbers.*

By reduction $x = \frac{14y - 11}{19} = wh.$ Alſo $\frac{19}{19}y = wh.$ Then by ſubtraction, $\frac{19y}{19} - \frac{14y - 11}{19} = \frac{5y + 11}{19} = wh.$ And multiplying by 4, $\frac{20y + 44}{19} = \frac{20y + 6}{19} + 2 = wh.$ And $\frac{20y + 6}{19} = wh.$ Subtract $\frac{19y}{19}$; and $\frac{y + 6}{19} = wh. = p.$ Whence $y = 19p - 6$. Let $p = 1$, for the leaſt affirmative value of y, and $y = 13$. Whence $x = 9$.

Or thus,

$y = \frac{19x + 11}{14} = x + \frac{5x + 11}{14} = wh.$ Then $\frac{5x + 11}{14} = wh.$ And multiplying by 3, $\frac{15x + 33}{14} = wh.$ But $\frac{14x + 28}{14} = wh.$ And ſubtracting, $\frac{x + 5}{14} = wh. = p.$ And $x = 14p - 5$. Let $p = 1$, to have x the leaſt; and $x = 9$, and $y = 13$.

Ex. 2.

Suppoſe $3x = 8y - 16$, *query* x, y.

Here $x = \frac{8y - 16}{3} = 2y - 5 + \frac{2y - 1}{3} = wh.$ And $\frac{2y - 1}{3} = wh.$ And multiplying by 2, $\frac{4y - 2}{3} = wh.$ But $\frac{3y}{3} = wh.$ And their difference $\frac{y-2}{3} = wh. = p$. Whence $y = 3p + 2$, and taking $p = 0$, $y = 2$. Whence $x = 0$.

Ex. 3.

Let $24x = 13y + 16$.

Here $x = \frac{13y + 16}{24} = wh.$ multiply by 11, and $\frac{143y + 176}{24} = wh.$ But $\frac{6 \times 24y + 7 \times 24}{24}$ or $\frac{144y + 168}{24} = wh.$ From which ſubtract the former, and $\frac{y-8}{24} = wh. = p$. And then $y = 24p + 8$, and putting $p = 0$, $y = 8$, and $x = 5$.

Ex. 4.

Let $14x = 4y + 7$.

Then $x = \frac{4y + 7}{14} = wh.$ And multiplying by 7, $\frac{28y + 49}{14}$ or $\frac{28y + 7}{14} + 3 = wh.$ And $\frac{28y + 7}{14} = wh.$ But $\frac{28y}{14} = wh.$ Therefore their difference $\frac{7}{14} = wh.$ which is abſurd; for an even number cannot divide an odd number, nor a greater number a leſſer. See Cor. 2. Prop. VIII. B. II. Arithmetic.

Ex. 5.

Let $27x = 1600 - 16y$.

Here $x = \frac{1600 - 16y}{27} = wh.$ abridged $\frac{7 - 16y}{27} = wh.$ or $\frac{16y - 7}{27} = wh.$ Subtract it from $\frac{27y}{27}$, and $\frac{11y+7}{27} = wh.$ multiply by 2, and $\frac{22y+14}{27} = wh.$ Subtract it from $\frac{27y+27}{27}$, and $\frac{5y+13}{27} = wh.$ multiply by 2, and $\frac{10y+26}{27} = wh.$ ſubtract it from $\frac{11y+7}{27}$, and $\frac{y-19}{27} = wh. = p$, and $y = p \times 27 + 19$, and if $p = 0$, $y = 19$, and $x = 48$.

Cor. 1. *All the values of* y *are had, by continually adding the coefficient of* x; *as* y, $y+a$, $y+2a$, $y+3a$, &c. *And all the values of* x *are had, by continually adding the coefficient of* y; *as* x, $x+b$, $x+2b$, &c; *or by ſubtracting them, for negative numbers, and both are in arithmetical progreſſion.*

Cor. 2. *When the proceſs brings out an odd number divided by an even number, or a leſſer number divided by a greater, which ſhould be a whole number; the queſtion is impoſſible.*

Cor. 3. *If it be required to find* y *a whole number, ſo that the fraction* $\frac{by+c}{a}$ *may alſo be a whole number. You muſt proceed the very ſame way, by abridging the fraction to* $\frac{y \pm g}{a}$, *and then find* $y = aP \mp g$, *where* P *is any whole number, taken at pleaſure.*

PROBLEM LXXVI.

To find ſuch a whole number x, *that being divided by the given numbers* a, b, c, &c. *ſhall leave the given remainders* f, g, h, &c.

RULE.

Since the fractions $\frac{x-f}{a}$, $\frac{x-g}{b}$, $\frac{x-h}{c}$, &c. are whole numbers; put the firſt $\frac{x-f}{a} = \text{P} = wh.$ Then $x = a\text{P} + f$. Put this value of x in the ſecond fraction; then $\frac{a\text{P}+f-g}{b} = wh.$ Then (Cor. 2. laſt Prob.) find $\text{P} = b\text{Q} + m$, where $\text{Q} = wh.$ then will $x = ab\text{Q} + am + f$. Put this value of x in the third fraction; then $\frac{ab\text{Q}+am+f-h}{c} = wh.$ Then, as before, find $\text{Q} = c\text{R} + n$; and put this inſtead of Q in the laſt value of x; then this value of x muſt be put into the fourth fraction; and proceed the ſame way through all the fractions. This is the method of proceeding; but numbers muſt be uſed all along inſtead of the ſmall letters. And the leaſt *wh.* number R may be taken at pleaſure.

Ex. 1.

To find a number which divided by 3, 5, 7, *and* 2; *will leave the remainders* 2, 4, 6, 0, *reſpectively.*

Let the number be x, then $\frac{x-2}{3}$, $\frac{x-4}{5}$, $\frac{x-6}{7}$ and $\frac{x-0}{2}$ are whole numbers. Let $\frac{x-2}{3} = \text{P}$, and $x = 3\text{P} + 2$; then $\frac{x-4}{5} = \frac{3\text{P}+2-4}{5} = \frac{3\text{P}-2}{5} = wh.$ ſubtract it from $\frac{5\text{P}}{5}$, and $\frac{2\text{P}+2}{5} = wh.$ Subtract this from $\frac{3\text{P}-2}{5}$; then $\frac{\text{P}-4}{5} = wh. = \text{Q}$, and $\text{P} = 5\text{Q} + 4$, and $x = 15\text{Q} + 14$. Again $\frac{x-6}{7} = \frac{15\text{Q}+8}{7} = wh.$ and $\frac{\text{Q}+1}{7} = wh. = \text{R}$, and $\text{Q} = 7\text{R} - 1$, and $x = 105\text{R} - 1$.

Laſtly $\frac{x-0}{2} = \frac{105R-1}{2} =$ *wh.* and $\frac{R-1}{2} =$ *wh.* $= S$, and $R = 2S + 1$. Whence $x = 210S + 104$, the number ſought; and putting $S = 0$, the leaſt value of x is 104.

Ex. 2.

To find a whole number, which being divided by 16, 17, 18, 19, 20; *will leave* 6, 7, 8, 9, 10, *remainders.*

Let $x =$ number. Then $\frac{x-6}{16}$, $\frac{x-7}{17}$, $\frac{x-8}{18}$, $\frac{x-9}{19}$, $\frac{x-10}{20}$ are whole numbers. Put $\frac{x-6}{16} = P$, then $x = 16P + 6$.

Then $\frac{x-7}{17} = \frac{16P-1}{17} =$ *wh.* And thence $\frac{P+1}{17} =$ *wh.* $= Q$, and $P = 17Q - 1$, and $x = 272Q - 10$.

Alſo $\frac{x-8}{18} = \frac{272Q-18}{18} =$ *wh.* and $\frac{2Q}{18} =$ *wh.* $= R$, and $Q = 9R$, whence $x = 2448R - 10$.

Again $\frac{x-9}{19} = \frac{2448R-19}{19} =$ *wh.* and $\frac{-3R}{19} =$ *wh.* or $\frac{3R}{19} =$ *wh.* and $\frac{18R}{19} =$ *wh.* whence $\frac{R}{19} =$ *wh.* $= S$, and $R = 19S$. Then $= 46512S - 10$.

Laſtly $\frac{x-10}{20} = \frac{46512S-20}{20} =$ *wh.* and $\frac{12S}{20} =$ *wh.* $= T$, and $S = 5T$. Whence $= 232550T - 10$. And if $T = 1$, then the eaſt value of $x = 232550$.

Ex.

Ex. 3.

To find a number (x)*, which being divided by* 3, 7, 14, 20; *there ſhall remain* 1, 3, 7, 14.

Here $\frac{x-1}{3}$, $\frac{x-3}{7}$, $\frac{x-7}{14}$, $\frac{x-14}{20}$ are whole numbers. Let $\frac{x-1}{3} = \text{P}$, and $x = 3\text{P}+1$.

Then $\frac{x-3}{7} = \frac{3\text{P}-2}{7} = wh.$ and $\frac{6\text{P}-4}{7} = wh.$ Whence $\frac{\text{P}+4}{7} = wh. = \text{Q}$, and $\text{P} = 7\text{Q}-4$, and $x = 21\text{Q}-11$.

Alſo $\frac{x-7}{14} = \frac{21\text{Q}-18}{14} = wh.$ and $\frac{7\text{Q}-4}{14} = wh.$ and $\frac{14\text{Q}-8}{14} = wh.$ Whence $\frac{8}{14} = wh.$ which is abſurd.

Hence the queſtion is impoſſible for the three firſt ſuppoſitions; but will hold good for two of them: in which caſe $x = 21\text{Q}-11$, where the leaſt value of x is 10.

2 RULE.

When two diviſors and their remainders are given; then find two fixed multipliers M, N: ſuch, that dividing them,

$\frac{\text{M}}{a}$ leaves 0, and $\frac{\text{M}}{b}$ leaves 1 remaining.

and $\frac{\text{N}}{a}$ leaves 1, and $\frac{\text{N}}{b}$ leaves 0 remaining.

Then divide $\frac{\text{M}g+\text{N}f}{ab}$, and the remainder is x, the number ſought.

Likewiſe for three diviſors and remainders; find three fixed multipliers M, N, P; ſuch, that by dividing them,

$\frac{M}{a}$ leaves 1, and $\frac{M}{bc}$ leaves 0, remaining.
$\frac{N}{b}$ leaves 1, $\frac{N}{ac}$ leaves 0, remaining.
$\frac{P}{c}$ leaves 1, $\frac{P}{ab}$ leaves 0, remaining.

Then dividing $\frac{Mf + Ng + Ph}{abc}$, the remainder is x, required; and the like for more quantities.

To prove the truth of this. Since (Caſe 1) $\frac{M}{a}$ as alſo $\frac{N}{b}$ leave 0, by diviſion; therefore $\frac{Mg}{a}$, and $\frac{Nf}{b}$ leave 0.

And ſince $\frac{M}{b}$, as alſo $\frac{N}{a}$ leave 1. Therefore $\frac{M - 1}{b}$ and $\frac{N - 1}{a}$ leave 0. Therefore $\frac{Mg - g}{b}$ and $\frac{Nf - f}{a}$ leave 0; that is, $\frac{Mg}{b}$ leaves g, and $\frac{Nf}{a}$ leaves f. Therefore $\frac{Mg + Nf}{a}$ leaves $0 + f$, and $\frac{Mg + Nf}{b}$ leaves $g + 0$.

But ſince $Mg + Nf$ may exceed ab, and therefore is not the leaſt number; therefore divide by ab, and the remainder is the leaſt number required. And the ſame way, Caſe 2, or any other, is proved.

Ex.

Ex. 4.

Having the cycle of the dominical letter f, and cycle of the moon g; to find the year of the Dionyſian *period.*

Let x be the year ſought. Then $\frac{x-f}{28}$ and $\frac{x-g}{19}$ are whole numbers. Here $a=28$, and $\frac{M}{28}=wh.=P$, and $M=28P$. Alſo $\frac{M-1}{19}=wh.$ $=\frac{28P-1}{19}$; and multiplying by 2, $\frac{56P-2}{19}$ $=wh.$ Alſo $\frac{57P}{19}=wh.$ Therefore $\frac{P+2}{19}=wh.$ $=Q$, and $P=19Q-2$. Whence $M=28\times\overline{19Q-2}$ $=532Q-56$, and if $Q=1$; then $M=476$.

Then $\frac{N}{b}=\frac{N}{19}=wh.=P$, and $N=19P$. Alſo $\frac{N-1}{28}=\frac{19P-1}{28}=wh.$ multiply by 3; then $\frac{57P-3}{28}=wh.$ and $\frac{56P}{28}=wh.$ therefore $\frac{P-3}{28}$ $=wh.=Q$, and $P=28Q+3$. Whence $N=28\times 19Q+57$, and if $Q=0$, $N=57$.

Therefore $x=$ remainder of $\frac{476g+57f}{532}$, which ſerves in general, for any numbers, f, g.

Let $f=10$, $g=12$; then $x=430$.

Ex. 5.

Having the cycle of the Sunday *letter f, the golden number g, and indiction h; to find the year of the* Julian *period.*

Here $a=28$, $b=19$, $c=15$, $ab=532$, $ac=420$, $bc=285$, and $abc=7980$.

Then $\frac{M}{285} = wh. = P$, and $M = 285P$. Alſo $\frac{M-1}{28} = \frac{285P-1}{28} = wh.$ This at laſt gives $P = 17$ the leaſt; and then $M = 4845$.

Again, $\frac{N}{420} = wh. = P$, and $N = 420P$. Alſo $\frac{N-1}{19} = \frac{420P-1}{19} = wh.$ and $\frac{2P-1}{19} = wh.$ which will give $P = 10$, and $N = 4200$.

Laſtly, $\frac{P}{532} = wh. = Q$, and $P = 532Q$. Alſo $\frac{P-1}{15} = \frac{532Q-1}{15} = wh.$ and $\frac{7Q-1}{15} = wh.$ at laſt $Q = 13$, and $P = 6916$. Whence the remainder of $\frac{4845f + 4200g + 6916h}{7980}$ is $= x$.

Let $f = 0$, or 28, $g = 1$, $h = 2$; then $x = 2072$.

Cor. *When the operation brings out a leſſer number divided by a greater, inſtead of a whole number; the problem is impoſſible.*

PROBLEM LXXVII.

An equation being given, containing ſeveral unknown quantities; to find their limits.

When an equation contains ſeveral unknown quantities, the values of all of them, except one, may be taken at pleaſure; and when their values are aſſigned, and numbers put for them in the equation, that ſingle quantity may alſo be found, by reducing the equation. And ſuch equations will admit of an infinite number of ſolutions, if we admit of fractional and negative numbers. But ſince theſe ſolutions are moſt uſeful where affirmative quantities are concerned; and more uſeful ſtill, when only affirmative whole numbers are admitted; therefore I propoſe to conſider only theſe two caſes, and particularly the laſt: becauſe in that caſe ſuch an equation will have a determined number of ſolutions. And therefore it is neceſſary to know the limits of the unknown quantities; leſt we go about to ſeek their values beyond theſe limits.

RULE.

Tranſpoſe the negative quantities to the contrary ſide; that all the terms may be affirmative. Then to find the limits of any one, put all the reſt $= 0$, or ſuppoſe them to vaniſh; and from hence find the value of that quantity, which will be one limit thereof. And to know which limit it is, conceive the other quantities to increaſe and have ſome certain value; then if by this, the value (of the unknown quantity under conſideration) increaſes; it is the leaſt limit you found; if it decreaſes, it is the greateſt limit. And in caſe you find no leaſt limit, then 0 is its leaſt limit. This proceſs relates to fractional quantities.

But if you only deſire whole numbers; put 1 for each of the other quantities, which is the leaſt value they can have; then from the reſulting equation, find your unknown quantity and its limit, as before directed.

Proceed the ſame way with all the unknown quantities.

Ex. 1.

Let $3a + 5e = 28$, *to find the limits of* a, e.

Let $e = 0$, then $3a = 28$, and $a = \frac{28}{3} = 9\frac{1}{3}$.

Now let e be ſome real quantity; it is plain the greater e is, the leſs a muſt be; therefore $9\frac{1}{3}$ is the greater limit. Whence $a \sqsupset 9\frac{1}{3}$.

For e: let $a = 0$, then $5e = 28$, and $e = \frac{28}{5} = 5\frac{3}{5}$. But if a increaſes, e decreaſes; therefore $5\frac{3}{5}$ is the greater limit, and $e \sqsupset 5\frac{3}{5}$, and the leſſer limit of both a and e, is 0. All this including fractions.

For whole numbers.

Let $e = 1$, then $3a = 28 - 5 = 23$, and $a = \frac{23}{3} = 7\frac{2}{3}$, the leſſer limit, and $a \sqsupset 7\frac{2}{3}$.

Again, let $a = 1$, then $5e = 25$, and $e = \frac{25}{5} = 5$, the greater limit, and $e =$ or $\sqsupset 5$.

Ex. 2.

Let $3a - 5e = 28$, *to find the limits of* a, e *in whole numbers.*

Then $3a = 28 + 5e$; let $e = 1$, then $3a = 33$, and $a = \frac{33}{3} = 11$; but when e increaſes a increaſes; therefore 11 is the leſſer limit, and $a =$ or $\sqsubset 11$.

Let $a = 1$, $28 + 5e = 3$, and e will be negative, which we exclude. But whilſt a increaſes e increaſes; therefore 0 is the leaſt limit of e, or $e \sqsubset 0$: and it has no greateſt limit.

Ex. 3.

Let $3x + 5y + 8z = 10003$, *to find the limits in whole numbers.*

Suppoſe $y = 1 = z$. Then $3x = 10003 - 13$, and $x = \frac{9990}{3} = 3330$. And ſince x decreaſes, whilſt y and z increaſe; therefore 3330 is the greater limit, and $x =$ or $\sqsupset 3330$.

Again, let $x = z = 1$; then $5y = 9992$, and $y = \frac{9992}{5} = 1998\frac{2}{5}$; and y decreaſes whilſt x, z increaſe: whence $y \sqsupset 1998\frac{2}{5}$.

Laſtly, for z; let $x = y = 1$, then $8z = 10003 - 8 = 9995$, and $z = \frac{9995}{8} = 1249\frac{3}{8}$. But z decreaſes, whilſt x, y increaſe; therefore $z \sqsupset 1249\frac{3}{8}$.

Ex. 4.

Let $13x - 5y + 8z = 10$, *to find the limits of* y.

Here $13x + 8z = 10 + 5y$; let $x = 1 = z$; then $5y + 10 = 21$, and $5y = 11$, and $y = 2\frac{1}{5}$; and whilſt x and z increaſe, y increaſes; therefore $2\frac{1}{5}$ is the leaſt limit, and $y \sqsubset 2\frac{1}{5}$.

Note, the limits of x and z cannot be found till the value of y be aſſigned.

PROBLEM LXXVIII.

Two equations being given, containing three or more unknown quantities; to determine their limits

RULE.

Having pitched upon the quantity you would limit; expunge one of the other quantities, and you will have one limiting equation. Then expunge another of them, and this gives another limiting equation. By theſe two equations find the limits of the quantity pitched on ſeparately, by the laſt problem.

But note, In any limiting equation, all the other unknown quantities therein, (being put on the ſame ſide of the equation, with the abſolute number,) muſt have the ſame ſign: otherwiſe, (if they have different ſigns) they cannot limit the quantity propoſed, till the value of ſome of the reſt be known.

If there be more equations, the proceſs is the ſame with any of them.

Ex. 1.

Let $a + e + y = 56$.
and $32a + 20e + 16y = 1232$; to limit a.

Multiply the firſt equation by 20, produces $20a + 20e + 20y = 1120$. Subtract this from the ſecond, and you have $12a - 4y = 112$: whence (Prob. lxxvii.) $a \sqsubset 9\frac{2}{3}$.

Multiply the firſt equation by 16, gives $16a + 16e + 16y = 896$. Subtract it from the ſecond, and $16a + 4e = 336$; whence $a \sqsupset 20\frac{3}{4}$.

In like manner, to limit y, multiply the firſt equation by 32, and $32a + 32e + 32y = 1792$. Subtract the ſecond from it, and $12e + 16y = 560$. This gives $y \sqsupset 34\frac{1}{4}$. And the equation $12a - 4y = 112$, gives $y \sqsubset 0$.

To limit e; the equation $16a + 4e = 336$, gives $e \sqsupset 80$. And the equation $12e + 16y = 560$, gives $e \sqsupset 45\frac{1}{3}$. But here is no leſſer limit for e; therefore $e \sqsubset 0$, and $\sqsupset 45$.

Ex. 2.

$$\text{Let } \left.\begin{aligned} 3x - y + 2u &= 20 \\ \text{and } 12x + 6y + 5u &= 150 \end{aligned}\right\}.$$

$$\begin{array}{llcl} \text{1ſt} \times 5 \text{ is,} & 15x - 5y + 10u & = & 100 \\ \text{2d} \times 2 \text{ is,} & 24x + 12y + 10u & = & 300 \\ \text{difference} & 9x + 17y & = & 200 \end{array}$$

This equation gives $x \sqsupset 20\frac{1}{3}$, and $y \sqsupset 11\frac{4}{17}$.

1ſt × 6 is, $18x - 6y + 12u = 120$; add this to the ſecond: then $30x + 17u = 270$,

whence $x \sqsupset 8\frac{13}{30}$, and $u \sqsupset 14\frac{2}{17}$.

1ſt × 4 is $12x - 4y + 8u = 80$. Subtract from the ſecond, $10y - 3n = 70$.

And $y \sqsubset 7\frac{3}{10}$, and $u \sqsubset 0$.

There is no leaſt limit for x; therefore $x \sqsubset 0$, and $\sqsupset 8\frac{13}{30}$.

Ex. 3.

Let $a + e + y + u = 100.$
and $16a + 10e + 8y + 6u = 1200.$

	1	$a + e + y + u = 100$
	2	$16a + 10e + 8y + 6u = 1200$
$1 \times (6)$	3	$6a + 6e + 6y + 6u = 600$
$2 - 3$	4	$10a + 4e + 2y = 600$
$4 \div$	5	$a = \frac{600-4-2}{10} = \frac{594}{10} = 59\frac{4}{10}$ at moſt
5	6	$a \sqsupset 59\frac{4}{10}.$
$1 \times (10)$	7	$10a + 10e + 10y + 10u = 1000$
$2 - 7$	8	$6a - 2y - 4u = 200$
8 tr.	9	$6a = 200 + 2y + 4u$
$9 \div (6)$	10	$a = \frac{206}{6} = 34\frac{1}{3}$ at leaſt, and $a \sqsubset 34\frac{1}{3}$; ſo a is between $34\frac{1}{3}$ and $59\frac{2}{5}$. Then for the other quantities.
$1 \times (8)$	11	$8a + 8e + 8y + 8u = 800.$
$1 \times (16)$	12	$16a + 16e + 16y + 16u = 1600.$
$2 - 11$	13	$8a + 2e - 2u = 400.$ This equation will limit u but not e. Here $u \sqsubset 0$.
$12 - 2$	14	$6e + 8y + 10u = 400.$
14,	15	$u = \frac{400-14}{10} = \frac{386}{10} = 38\frac{6}{10}$, or $u \sqsupset 38\frac{6}{10}$
4,	16	$y = \frac{600-4e-10a}{2} = \frac{586}{2} = 293$, or $y \sqsupset 293$; but, ſince the limits of a, are known ; y may be determined more exactly ; thus $y = \frac{596-10\times 35}{2} = \frac{246}{2} = 123$, or $y \sqsupset 123.$

14	17	Again $y = \frac{400-16}{8} = \frac{384}{8} = 48$, or $y =$ or $\sqsupset 48$. But theſe are all greater limits of y, and there wants the leſſer limit; therefore $y \sqsubset 0$, and $\sqsupset 48$.
14	18	$e = \frac{400-18}{6} = \frac{382}{6} = 63\frac{2}{3}$, or $e = \sqsupset 63\frac{2}{3}$, But the leaſt limit of e cannot be found; therefore take $e \sqsubset 0$.

SCHOLIUM.

When three numbers are ſought by two equations; all the values of each of them, in whole numbers, make three ſeries of arithmetical progreſſion, taken within the limits of theſe numbers. And if four or more numbers are ſought, the value of each is to be found in ſeveral arithmetical progreſſions. But yet the values of any three will be in arithmetic progreſſion, when the values of all the reſt are aſſigned, as before for three numbers.

For in the caſe of three numbers, and two equations; any one of the three may be expunged; and then you will have but one equation, and two unknown quantities; which brings it under Prob. lxxv. But by Cor. 1. of that problem, theſe two remaining quantities are contained in two ſeries of arithmetical progreſſion. And as any of the three may be expunged; therefore any two of them will conſtitute two ſeries of arithmetical progreſſion.

PROBLEM LXXIX.

The prices of ſeveral ingredients being given, to find the quantities thereof; ſo that the mixture may be ſold at a given price.

Suppoſe four ſimples A, B, C, D, are to be mixed; and their prices to be as follows:

$$\begin{array}{rcl} \text{Mean price} & = & m \\ \text{Price of A} & = & m+a \\ \text{of B} & = & m+b \\ \text{of C} & = & m-c \\ \text{of D} & = & m-d. \end{array}$$

And let the quantities to be taken of A, B, C, D, be x, y, z, v, reſpectively. Place them in order, thus:

$$m\left\{\begin{array}{c|c} \textit{prices} & \textit{quantities} \\ m+a & x \\ m+b & y \\ & \\ m-c & z \\ m-d & v \end{array}\right.$$

Then by the nature of the queſtion; if each quantity be multiplied by its price, the ſum of the products will be equal to the ſum of all the quantities multiplied by the mean price; that is,

$$\left.\begin{array}{l} \overline{m+a}\times x+\overline{m-d}\times v \\ +\overline{m+b}\times y+\overline{m-c}\times z \end{array}\right\}=\overline{x+v+y+z}\times m.$$

$$\text{Let } \overline{m+a}\times x+\overline{m-d}\times v=\overline{x+v}\times m.$$
$$\text{And } \overline{m+b}\times y+\overline{m-c}\times z=\overline{y+z}\times m.$$

That is,

$$mx+ax+mv-dv=mx+mv$$
$$my+by+mz-cz=my+mz.$$

by the former, $ax-dv=0$, or $ax=dv$.
by the latter, $by-cz=0$, or $by=cz$.

Now ſince x and y may be taken at pleaſure. Therefore put $x=d$, and $y=c$. Then will $v=a$, and $z=b$. Whence the quantities will be ranged thus:

$$m\begin{cases} m+a \\ m+b \\ \\ m-c \\ m-d \end{cases} \left| \begin{array}{ll} d & \\ & c \\ & \\ & b \\ a & \end{array} \right.$$

which gives this

RULE.

Couple every greater rate with one leſſer than the mean price ($m+a$ and $m-d$; alſo $m+b$ and $m-c$); then take the difference between each rate and the mean rate, and place it *alternately*, that is, againſt the quantity it is coupled with; do the ſame with all the rates, (thus place a againſt $m-d$, b againſt $m-c$, c againſt $m+b$, d againſt $m+a$); then if none of the quantities of A, B, C, D, be given. Then d, c, b, a will be the quantities of each to be taken for the mixture. But if any one quantity be given; then all the quantities d, c, b, a muſt be increaſed or decreaſed in proportion. Or if the ſum of the quantities be given, then other quantities muſt be taken in proportion, ſo that $d+c+b+a$ may be to the ſum given, as any of the differences d, c, &c. to the reſpective quantity required. And this is the common rule of *Alligation Alternate.*

Again,

Since $ax = dv$, and $by = cz$. Take $x = md$, and $y = nc$; then $v = ma$, and $z = nb$. Then putting md, nc, nb, ma, for x, y, z, v reſpectively; and the caſe will ſtand thus:

$$m\begin{cases} m+a \\ m+d \\ \\ m-c \\ m-d \end{cases} \left| \begin{array}{ll} md & \\ & nc \\ & \\ & nb \\ ma. & \end{array} \right.$$

which gives this

RULE.

Having coupled the rates as before directed, and taken the differences. Then inſtead of any couple of the differences, you may take any equimultiples thereof; and place them *alternately.* And theſe (or other quantities proportional to them), will be the quantities required. And this is the *Rule of Alligation* improved.

PROBLEM LXXX.

If the numbers A *and* B *be produced from a and b, by any ſimilar operation; to find the number from which* N *is produced, by the like operation. Suppoſing the differences of the numbers* A, B, N, *to be as the differences of a, b, and the unknown number.*

$a \quad b \quad z$
$A \quad B \quad N$

Let z be the number ſought, and put the differences $N - A = r$, $N - B = s$. Then by the queſtion, $r\ (N - A) : s\ (N - B) :: z - a : z - b$. Then $rz - rb = sz - sa$. And by tranſpoſition, $rz - sz = rb - sa$, and $z = \frac{rb - sa}{r - s}$; or if s be negative (or B greater than N), then $z = \frac{rb + sa}{r + s}$, the number ſought.

Cor. 1. *Hence is derived the practice of the double Rule of Falſe. For if both* A *and* B *be leſſer than* N, *or both greater; then* $z = \frac{rb - sa}{r - s}$. *But if only one as* B *be greater than* N, *then s is negative, and* $z = \frac{rb + sa}{r + s}$.

That is, if each ſuppoſed number be multiplied by the error of the other, and the difference of the products be divided by the difference of the errors, when the errors are like; or the ſum of the products divided by the ſum of the errors, when the errors are unlike; the quotient gives the number ſought.

Cor. 2. *Hence alſo is derived another method of orking the Rule of Approximation, or Rule of Falſe, hich is this.*

Multiply the difference of the ſuppoſed numbers, by e leaſt error, and divide the product, by the difrence of the errors, if like; or by the ſum if unlike. he quotient is the correction of the number belonging the leaſt error.

Then this correction is to be added or ſubtracted, ccording as that number was too little or too great.

For let s be the leaſt error, being the error of , and $q =$ the correction; then if A, B be leſs han N, $b+q=z$, and $q=z-b=\dfrac{rb-sa}{r-s}-b=$

$$\frac{b-sa-rb+sb}{r-s}=\frac{b-a}{r-s}s.$$

But if B is greater than N, then $b-q=z$, and

$$=b-z=b-\frac{rb+sa}{r+s}=\frac{rb+sb-rb-sa}{r+s}=$$

$$\frac{-a}{+s}s.$$

Scholium.

Since it has been ſhewn, that the number ſought vill come out exactly, by this rule, when the errors are exactly proportional to the differences of he ſuppoſed numbers from the true one. Thereore it follows, that when the errors are *nearly* proportional to theſe differences, that the anſwer will come out *nearly* true. And theſe proportions will be the nearer to an equality, the nearer theſe ſuppoſed numbers are taken to the true number. And therefore in all queſtions where this rule is applied, every operation will bring us nearer the true anſwer, if we always take the neareſt numbers, (where the errors are leaſt) for new ſuppoſitions. And thus repeating the operation, one may continually approximate to the true number, within any degree of exactneſs required; let the particular queſtion be of what nature it will.

Upon this rule alſo is founded the rule of finding proportional parts.

PROBLEM LXXXI.

Suppoſe A, B, C, D, *&c. to be ſeveral ſorts of goods; and* m, n, p, q, &c. *given numbers; and the values of theſe goods are*

$$\begin{aligned} mA &= nB \\ pB &= qC \\ rC &= sD \\ tD &= vE \end{aligned}$$

To find what quantity of the laſt ſort is equal to a given quantity of the firſt: and the reverſe.

Let z times the laſt be $=y$ times the firſt, that is, let $zE = yA$.

Multiply all theſe equations together; the firſt ſide by the firſt, and the ſecond by the ſecond. Then we have $mA \times pB \times rC \times tD \times zE = nB \times qC \times sD \times vE \times yA$. Then $mprtz = nqsvy$. Then if the quantity of the laſt ſort be required, $z = \frac{nqsvy}{mprt}$. But if the quantity of the firſt ſort be ſought; $y = \frac{mprtz}{nqsv}$. Whence this

RULE.

Place the terms in two columns, ſo that there may not be two terms of a ſort in either column. Then multiply the numbers in the leſſer column for a diviſor; and the numbers in the greater column (with the odd term) for a dividend. The quotien quotient is the quantity of that ſort which ſtands ingle in the two columns. And this is the *Rule f Exchange* in arithmetic.

PROBLEM LXXXII.

To inveſtigate numbers for rational ſquares, cubes, &c.

Problems of this ſort are often capable of an nfinite number of anſwers; and yet none of the quantities can be aſſumed at pleaſure, but muſt be inveſtigated as follows.

RULE.

Put one or more letters to denote the root of the ſquare, cube, *&c.* Which letters muſt be ſo aſſumed, that when the equation is involved, either the given number, or the higheſt power of the unknown quantity, may be on both ſides of the equation, and conſequently vaniſhes out of it. And then if the unknown quantity be but of one dimenſion, the problem is ſolved, by reducing the equation. But if the unknown quantity is ſtill a ſquare or higher power; you muſt farther aſſume other new letters, to denote the root, and proceed as before; till you get the unknown quantity of one dimenſion; and from this unknown quantity all the reſt are to be determined. For the whole art is, ſo to denote the root of the given power, that the unknown quantity may be reduced to one dimenſion.

But no general rule of proceeding can be given to ſuit all caſes; and therefore the ſolution will often be left to the ſagacity of the analyſt, in contriving ſuch a deſignation of letters as is proper for the purpoſe.

Ex. 1.

To find two ſuch numbers, ſo that the ſum of thei ſquares is a ſquare.

Let x, y, z be the roots of the ſquares, ſo tha $xx+yy=zz$. Aſſume $z=y+r$, then $xx+yy=zz$ $=yy+2ry+rr$, and $xx=2ry+rr$, and $2ry=xx-rr$ where y the unknown quantity is of one dimenſion, which reduced gives $y=\frac{xx-rr}{2r}$; and $y+r=\frac{xx-rr}{2r}+r=\frac{xx+rr}{2r}=z$. Therefor the numbers are x, $\frac{xx-rr}{2r}$ and $\frac{xx+rr}{2r}$, where and r denote any numbers taken at pleaſure.

But if the anſwer is required in whole numbers then $2rx$, $xx-rr$, $xx+rr$ will denote the roots o the ſquares, where the ſum of the two firſt is equa to the laſt ſquare.

Cor. *The three ſides of a right-angled triangle wil only be commenſurable, when* $xx+rr$ *denotes the hypothenuſe, and* $xx-rr$, *and* $2rx$ *the two ſides;* x, *being any numbers taken at pleaſure, ſo as* x *i greater than* r.

Ex. 2.

To find two numbers, the ſum of whoſe ſquares i equal to the ſum of two given ſquares.

Let x, y be the roots; aa, bb the given ſquares Aſſume $x=a-v$, $y=vz-b$. Then $xx+yy=a$ $+bb=aa-2av+vv+vvzz-2bvz+bb$; an $vv+vvzz=2av+2bvz$, and $v+vzz=2a+2bz$ and $v=\frac{2a+2bz}{zz+1}$. Where z is any number taken at pleaſure. Then $x=\frac{azz-2bz-a}{zz+1}$, an $y=\frac{2ax+bzz-b}{zz+1}$.

Or thus,

Let $x=a-v$, then $aa-2av+vv+yy=aa+bb$; and $yy-2av+vv=bb$. Put $y=vz-b$, then $vvzz-2bzv+bb-2av+vv=bb$, and $vvzz+vv=2bzv+2av$, or $vzz+v=2bz+2a$, and $v=\frac{2bz+2a}{zz+1}$, as before.

Ex. 3.

To find two numbers, ſuch that when either of them is added to the ſquare of the other, the ſum will be a ſquare number.

Let the numbers be x, y; then $xx+y=\square$, and $yy+x=\square$. Let $xx+y=\overline{r-x}^{2}=rr-2rx+xx$; then $y=rr-2rx$, and $2rx=rr-y$, whence $x=\frac{rr-y}{2r}$.

Again, aſſume $yy+x$ or $yy+\frac{rr-y}{2r}=\overline{y+v}^{2}=yy+2yv+vv$. Then $\frac{rr-y}{2r}=2yv+vv$, whence $rr-y=4ryv+2rvv$, and $4rvy+y=rr-2rvv$; whence $y=\frac{rr-2rvv}{4rv+1}$. And $x=\frac{2rrv+vv}{4rv+1}$, where r, v may be taken at pleaſure, provided r be greater than $2vv$.

Otherwiſe,

Since $x=\frac{rr-y}{2r}=\frac{1}{2}r-\frac{y}{2r}$, and $yy+x$ or $yy-\frac{y}{2r}+\frac{1}{2}r=\square$, put $yy-\frac{y}{2r}+\frac{1}{2}r=yy-\frac{y}{2r}+\frac{1}{16}rr=\overline{y-\frac{1}{4r}}^{2}$. Then $\frac{1}{2}r=\frac{1}{16rr}$, and $r^{3}=\frac{1}{8}$, which is a cube a cube number. And therefore will anſwer the queſtion; and we have $r = \frac{1}{2}$; whence $x = \frac{1}{4} - y$, and y may be any thing leſs than $\frac{1}{4}$.

Ex. 4.

To find two numbers in a given ratio, ſo that either of them added to the ſquare of the ſum, may make a ſquare.

Let the ratio of the two numbers be as b to c, and put $b + c = d$, and let the numbers be bx and cx. Then the ſquare of the ſum is $\overline{bx + cx}|^2 = ddxx$. Therefore $ddxx + bx = \square$, and $ddxx + cx = \square$.

Put $ddxx + bx = \overline{dx - v}|^2 = ddxx - 2dxv + vv$; then $bx = vv - 2dxv$, or $bx + 2dxv = vv$, and $x = \frac{vv}{b + 2dv}$. Then $ddxx + cx$ or $\overline{ddx + c} \times x = \frac{ddvv + bc + 2cdv}{b + 2dv} \times \frac{vv}{b + 2dv} = \square$, but $\frac{vv}{\overline{b + 2dv}|^2} = \square$; therefore $ddvv + bc + 2cdv = \square$ (See Cor. 27. II. Arithm.); aſſume $ddvv + bc + 2cdv = \overline{dv - z}|^2 = ddvv - 2dvz + zz$; then $2cdv + 2dzv = zz - bc$; and $v = \frac{zz - bc}{2cd + 2dz}$. Where zz muſt be greater than bc, and expunging v, $x = \frac{\overline{zz - bc}|^2}{4ddz \times \overline{b + z} \times \overline{c + z}}$.

SCHOLIUM.

It appears from theſe operations, that when a quantity, which is to be a ſquare by the problem, is not an algebraic ſquare; we muſt make it ſo, by aſſuming ſome new quantities to compleat it. Then theſe ſquares being compared, an equation is had for determining the unknown quantity. And in working, one may multiply or divide by any quantity which is a ſquare, and what is left will be a ſquare, in a more ſimple form. The like for other powers.

PROBLEM LXXXIII.

To determine the maximum or minimum of a quantity propoſed.

When a quantity is required to be the greateſt or leaſt poſſible, it is called a *maximum* or *minimum*. And at the time it becomes ſuch, it is at a ſtand, and at that moment neither increaſes nor decreaſes. Therefore to compute it.

RULE.

Calculate the value of the maximum or minimum two different ways, which is done by increaſing the unknown quantity therein, by an exceeding ſmall part; then theſe values are to be put equal to one another. The ſame muſt be done, if there be ſeveral variable quantities. But go no farther than the firſt power of the ſmall added part. Or,

If the maximum or minimum conſiſts of two parts; compute the exceeding ſmall increment of one, and the decrement of the other; and put them equal to one another.

Ex. 1.

What fraction is that whoſe ſquare exceeds its cube the greateſt poſſible.

Let x be the fraction, then $x^2 - x^3 = max.$ Take e an exceeding ſmall part to be added to x, then you will alſo have $\overline{x+e}\vert^2 - \overline{x+e}\vert^3 = max.$ that is, $xx + 2xe - x^3 - 3x^2e = max.$ Whence $x^2 - x^3 = xx + 2xe - x^3 - 3x^2e$, and $2xe - 3x^2e = 0$, or $3x^2e = 2xe$, and $3x = 2$, or $x = \frac{2}{3}$.

Or thus,

Since $x^2 - x^3 = max.$ let e be the ſmall increaſe of x, then $2xe$ is the increment of xx, and $3x^2e$ is the decrement of x^3; therefore $2xe = 3x^2c$, and $x = \frac{2}{3}$ as before.

Ex. 2.

To divide a given quantity into two parts, that one of the parts multiplied by the cube of the other part; the product may be a maximum.

Let a be the quantity, and x one part, and $a - x$ the other part, and e a ſmall additional part to x. Then $x^3 \times \overline{a - x}$ or $ax^3 - x^4 = max. = ax^3 + 3ax^2e - x^4 - 4x^3e.$ Then $3ax^2e = 4x^3e$, and $x = \frac{3}{4}a$, for one part, and $a - x = \frac{1}{4}a$, the other part.

Ex. 3.

To find $a^3 - a^2x + x^3$ a minimum, x being unknown.

Put $x + e$ for x. Then $a^3 - a^2x + x^3 = min. = a^3 - a^2x - a^2e + x^3 + 3x^2e$, and $-a^2e + 3x^2e = 0$, and $3xx = aa$; whence $x = a\sqrt{\frac{1}{3}}$. Then $a^3 - a^2x + x^3 = a^3 - a^3\sqrt{\frac{1}{3}} + \frac{1}{3}a^3\sqrt{\frac{1}{3}} = a^3 \times \overline{1 - \frac{2}{3}\sqrt{\frac{1}{3}}}$, the minimum.

Ex. 4.

Let $\frac{baax + aaxx - bx^3 - x^4}{baa + x^3} - a + x$ *be a maximum.*

This reduced to a common denominator is $\frac{2bbaax + aaxx - bx^3 - ba^3 - ax^3}{baa + x^3} = max.$ Put $x + e$ for x, Then $\frac{2baax + aaxx - bx^3 - ba^3 - ax^3}{baa + x^3} =$

$$\frac{2baax + 2baae + aaxx + 2aaxe - bx^3 - 3bx^2e - ba^3 - ax^3 - 3ax^2e}{baa + x^3 + 3xxe};$$

Then multiplying alternately, $\overline{2baax + aaxx - bx^3 - ba^3 - ax^3} \times \overline{baa + x^3 + 3xxe} = : 2baax + 2baae + aaxx + 2aaxe - bx^3 - 3bx^2e - ba^3 - ax^3 - 3ax^2e : \times \overline{baa + x^3}$. And throwing out what is common on both ſides, $\overline{2baax + aaxx - bx^3 - ba^3 - ax^3} \times 3xxe = \overline{baa + x^3} \times \overline{2baae + 2aaxe - 3bxxe - 3axxe}$. That is (dividing by e), $6baax^3 + 3aax^4 - 3bx^5 - 3ba^3xx - 3ax^5 = 2bba^4 + 2ba^4x - 3bbaaxx - 3ba^3xx + 2baax^3 + 2aax^4 - 3bx^5 - 3ax^5$. Reduced, $4baax^3 + aax^4 = 2bba^4 + 2ba^4x - 3bbaaxx$; or dividing by a, and tranſpoſing, $x^4 + 4bx^3 + 3bbxx - 2baax - 2bbaa = 0$.

Ex. 5.

Suppoſe $y^3 - 3yyx + 3yxx = nyx - nxx$.
and $x - \frac{1}{2}y = max$.

Suppoſe the maximum $= m$. Then $x = m + \frac{1}{2}y$. This ſubſtituted in the firſt equation, and reduced, gives $\frac{1}{4}y^3 + 3m^2y = \frac{1}{4}uyy - mmn$. And $y^3 + 12mmy - nyy + 4mmn = 0$. Where m is a fixt quantity. Put $y + e$ for y; then $y^3 + 12m^2y - nyy + 4m^2n = 0 = y^3 + 3y^2e + 12m^2y + 12m^2e - ny^2 - 2nye + 4m^2n = 0$, and $3y^2e + 12m^2e - 2nye = 0$, whence $3yy + 12m^2 - 2ny = 0$, or $2ny - 3yy = 12mm$. From this equation, and $y^3 + 12m^2y - nyy + 4m^2n = 0$, the quantities y and m will eaſily be determined.

Ex. 6.

Through a given point P *within the angle* BAC, *to draw a right line* BPC, *making the area of the triangle* BAC, *the leaſt poſſible.* Fig. 1.

Draw AP, and bPc extremely near BPC; then the area ABP + ACP = *minimum*. In the very

Fig. 1. ſmall triangles BPb and CPc, the vertical angles at P are equal, and $BP = bP$, as alſo $CP = cP$, extream near. Therefore the areas BPb, and CPc, are to one another as BP^2 to CP^2 (Geom. 19. II). But CPc is the increment of the area APC; and BPb is the decrement of the area APB. Therefore $BPb = CPc$, or $BP^2 = CP^2$; therefore $BP = CP$. Whence if PD be drawn parallel to CA, then $DB = DA$.

Ex. 7.

2. *To find the greateſt triangle inſcribed in a circle* ACBD.

Draw the diameter AB, and CD perpendicular thereto; alſo draw AC, AD. Let $AB = d$, $AE = x$, $EC = y$: then triangle $ACD = xy = max.$ or $xxyy = max.$ but $yy = dx - xx$: therefore $dx^3 - x^4 = max. = dx^3 + 3dx^2e - x^4 - 4x^3e$ (putting $x + e$ for x), and $3dx^2e = 4x^3e$, or $4x = 3d$, whence $x = \frac{3}{4}d$.

SCHOLIUM.

When any quantity is a maximum or minimum, its root, or its ſquare, or its cube, *&c.* will likewiſe be a maximum or minimum. Alſo when any quantity is a maximum or minimum, any given quantity may be added to it, or ſubtracted from it, and it will ſtill be a maximum, or minimum. Likewiſe it may be multiplied or divided by any given quantity, and ſtill remain a maximum or minimum.

PROBLEM LXXXIV.

A number or quantity being given; to find its logarithm by a ſeries, or to turn numbers into logarithms.

Let $\frac{x}{y}$ be the quantity given; $M = 1$, for *Neper*'s logarithms, or $M = ,434294482$, for the common logarithms. And let $x-y=v$, $x+y=z$. Then the logarithm of $\frac{x}{y}$, will be denoted theſe ſeveral ways following, deduced from the nature of logarithms.

$$\text{1. Log: } \frac{x}{y} = \mathrm{M}\times: \frac{v}{y} - \frac{v^2}{2y^2} + \frac{v^3}{3y^3} - \frac{v^4}{4y^4} + \frac{v^5}{5y^5} - \text{\&c.}$$

Or,

$$\text{2. Log: } \frac{x}{y} = \mathrm{M}\times: \frac{v}{x} + \frac{v^2}{2x^2} + \frac{v^3}{3x^3} + \frac{v^4}{4x^4} + \frac{v^5}{5x^5} + \text{\&c.}$$

Or,

$$\text{3. Log: } \frac{x}{y} = 2\mathrm{M}\times: \frac{v}{z} + \frac{v^3}{3z^3} + \frac{v^5}{5z^5} + \frac{v^7}{7z^7} + \text{\&c.}$$

Cor. 1. *If v be far leſs than* 1. *Then*

$$\text{Log: } \overline{1+v} = \mathrm{M}\times: v - \frac{vv}{2} + \frac{v^3}{3} - \frac{v^4}{4} + \frac{v^5}{5} \text{ \&c.}$$

This is plain by putting $y=1$. For then $x = 1+v$, and $\frac{x}{y} = 1+v$.

Cor. 2. $\text{Log: } y+v = \text{log: } y, + \mathrm{M}\times: \frac{v}{y} - \frac{v^2}{2y^2} + \frac{v^3}{3y^3} - \frac{v^4}{4y^4}$ &c.

or $\textit{log: } y+v = \textit{log: } y, + \mathrm{M}\times: \frac{v}{x} + \frac{v^2}{2x^2} + \frac{v^3}{3x^3} + \frac{v^4}{4x^4}$ &c.

or $\textit{log: } y+v = \textit{log: } y, + 2\mathrm{M}\times: \frac{v}{z} + \frac{v^3}{3z^3} + \frac{v^5}{5z^5} + \frac{v^7}{7z^7}$ &c.

For log: x or $y+v = \text{log: } y \times \frac{x}{y} = \text{log: } y + \text{log: } \frac{x}{y}$.

Cor. 3. *If $l =$ logarithm of n, and $l + s =$ logarithm of $n + v$. Then the additional part of the logarithm, that is,*

$$s = M \times : \frac{v}{n} - \frac{v^2}{2n^2} + \frac{v^3}{3n^3} - \frac{v^4}{4n^4} \ \&c.$$

or $$s = M \times : \frac{v}{n+v} + \frac{v^2}{2.\overline{n+v}|^2} + \frac{v^3}{3.\overline{n+v}|^3} \ \&c.$$

or $$s = 2M \times : \frac{v}{2n+v} + \frac{v^3}{3.\overline{2n+v}|^3} + \frac{v^5}{5.\overline{2n+v}|^5} \ \&c.$$

For ſince $l+s=$log: $\overline{n+v}$, and $l=$log: n; therefore $l + s - l =$ log: $\overline{n+v}$ — log: $n =$ log: $\frac{n+v}{n}$, that is, $s =$ log: $\frac{n+v}{n}$. And by this prop. (writing n for y, $n+v$ for x, and $2n+v$ for z); s or log: $\frac{n+v}{n}$ will come out as above.

Cor. 4. *If x be far leſs than a, then*

$$\text{Log: } a + bx + cx^2 + dx^3 \ \&c. = \text{log}: a + M \times : \frac{bx + cxx + dx^3 \ \&c.}{a} - \frac{\overline{bx + ccxx \ \&c.}|^2}{2aa} + \frac{\overline{bx \ \&c.}|^3}{3a^3} - \&c.$$

and log: $$\frac{a}{a - bx - cxx - dx^3 \ \&c.} = M \times : \frac{bx + cxx + dx^3 \ \&c.}{a} + \frac{\overline{bx + cxx \ \&c.}|^2}{2aa} + \frac{\overline{bx \ \&c.}|^3}{3a^3} \ \&c.$$

and log: $\frac{a + bx + cxx + dx^3 \text{ \&c.}}{a - bx - cxx - dx^3 \text{ \&c.}} = 2M \times :$

$$\frac{bx + cxx + dx^3 \text{ \&c.}}{a} + \frac{\overline{bx + cxx \text{ \&c}}|^3}{3a^3}$$

$$+ \frac{\overline{bx + \text{ \&c.}}|^5}{5a^5} \text{ \&c:}$$

The firſt caſe appears from Caſe 1, Cor. 2. writing $a + bx + cxx$ &c. for x, a for y, and $bx + cxx$ &c. for v.

The ſecond appears from Caſe 2. of this prop. writing a for x, $a - bx - cxx$ &c. for y, and $bx + cxx$ &c. for v.

The third appears from Caſe 3. of the prop. writing $a + bx + cxx$ &c. for x, $a - bx - cxx$ &c. for y, $2bx + 2cxx$ &c. for v, and $2a$ for z.

SCHOLIUM.

The $\log : y + v = \log : y : + \frac{2Mv}{2y + v}$ very near, when v is very ſmall; which is only the firſt term of the ſeries, Caſe 3. Cor. 2.

PROBLEM LXXXV.

A logarithm being given; to find the quantity belonging to it, or its number, by a ſeries. Or to turn logarithms into numbers.

Let $l + s$ be the logarithm given, $n + v$ its number, and let l be the logarithm of the number n.

Put $m = 2.302585093 = \frac{1}{M}$, for the common logarithms, or $m = 1$, for *Neper*'s logarithms. Then by Cor. 3. laſt Prob. $s = M \times : \frac{v}{n} - \frac{v^2}{2n^2} + \frac{v^3}{3n^3}$ &c.

and $\frac{s}{M}$ or $ms = \frac{v}{n} - \frac{v^2}{2n^2} + \frac{v^3}{3n^3}$ &c. Then by reverſion of ſeries (Prob. lxii.), $\frac{v}{n} = ms + \frac{\overline{ms}|^2}{2} + \frac{\overline{ms}|^3}{2.3} + \frac{\overline{ms}|^4}{2.3.4}$ &c. Then

1. $v = n \times : ms + \frac{\overline{ms}|^2}{2} + \frac{\overline{ms}|^3}{2.3}$ &c. Whence

2. $n + v = n \times : 1 + ms + \frac{\overline{ms}|^2}{2} + \frac{\overline{ms}|^3}{2.3} + \frac{\overline{ms}|^4}{2.3.4}$ &c. and

3. $\frac{n+v}{n} = 1 + ms + \frac{\overline{ms}|^2}{2} + \frac{\overline{ms}|^3}{2.3}$ &c.

That is,

Number of $l + s$ = number of $l \times : 1 + ms + \frac{\overline{ms}|^2}{2} + \frac{\overline{ms}|^3}{2.3} + \frac{\overline{ms}|^4}{2.3.4}$ &c.

Cor. 1. *If* $n = 1$, *and* $l = 0$; *then* $1 + v$ or number of $s = 1 + ms + \frac{\overline{ms}|^2}{2} + \frac{\overline{ms}|^3}{2.3} + \frac{\overline{ms}|^4}{2.3.4}$ &c.

Cor. 2. *If* $l =$ *log:* n, *and* $l + s =$ *log:* $n + v$; *then the additional part of the number, that is,*

$$v = n \times : ms + \frac{\overline{ms}|^2}{2} + \frac{\overline{ms}|^3}{2.3} + \frac{\overline{ms}|^4}{2.3.4} \text{ \&c.}$$

Cor. 3. *If* L *be the log: of the number* N, *then*

$$N^x = 1 + mxL + \frac{\overline{mxL}|^2}{2} + \frac{\overline{mxL}|^3}{2.3} + \frac{\overline{mxL}|^4}{2.3.4} \text{ \&c.}$$

For, by Cor. 1. $1 + v$ (numb. of s log.) $= 1 + ms + \frac{\overline{ms}|^2}{2} + \frac{\overline{ms}|^3}{2.3}$ &c. Where $1 + v$ may repreſent any number, and s its logarithm. Therefore let $1 + v = N$, and $s = L$; then

N (numb. of L log.) $= 1 + mL + \frac{\overline{mL}|^2}{2} + \frac{\overline{mL}|^3}{2.3}$ &c. therefore by the nature of logarithms,

N^x (numb. of xL log.) $= 1 + mxL + \frac{\overline{mxL}|^2}{2}$ $\frac{\overline{mxL}|^3}{2.3}$ &c.

Cor. 4. *If* $y = n + v$, $x = r + e$, $l = log: n$. *Then* y^x *or* $\overline{n + v}|^{r+e} = n^r \times : 1 + mel + \frac{\overline{mel}|^2}{2} + \frac{\overline{mel}|^3}{2.3}$ &c. $\times : 1 + \frac{r+e}{1} \times \frac{v}{n} + \frac{r+e}{1} \times \frac{r+e-1}{2} \times \frac{vv}{nn} + \frac{r+e}{1} \times \frac{r+e-1}{2} \times \frac{r+e-2}{3} \times \frac{v^3}{n^3}$ &c.

For $\overline{n + v}|^{r+e} = n^{r+e} \times \overline{1 + \frac{v}{n}}|^{r+e} = n^r \times n^e \times \overline{1 + \frac{v}{n}}|^{r+e}$, but by Cor. 3. $n^e = 1 + mel + \frac{\overline{mel}|^2}{2} + \frac{\overline{mel}|^3}{2.3}$ &c. and $\overline{1 + \frac{v}{n}}|^{r+e} = 1 + \frac{r+e}{1} \times \frac{v}{n} + \frac{r+e}{1} \times \frac{r+e-1}{2} \times \frac{vv}{nn}$ &c.

Cor. 5. *If* $l = log: n$; *then*

$N^{r+e} = n^r \times : 1 + mel + \frac{\overline{mel}|^2}{2} + \frac{\overline{mel}|^3}{2.3}$ &c.

For here $v = 0$.

Cor. 6. *If v, e be exceeding ſmall, then*

$\overline{n+v}^{r+e} = n^r \times : \overline{1 + mel + \frac{rv}{n}}$ *nearly, being only the firſt power of e and v.*

Cor. 7. *If n = number of the logarithm a; then the number of the logarithm $a + bx + cxx + dx^3$* &c.

$= n \times$ *into* $1 + m \times \overline{bx + cx^2 + dx^3 \text{ \&c.}} + \frac{m^2}{2} \times \overline{bx + cxx \text{ \&c.}}|^2 + \frac{m^3}{2.3} \times \overline{bx + cxx \text{ \&c.}}|^3 + \frac{m^4}{2.3.4} \times \overline{bx + \text{\&c.}}|^4 + \text{\&c.}$

This follows from this problem, putting $l = a$, and $s = bx + cxx$ &c.

PROBLEM LXXXVI.

A problem being reſolved analytically, to demonſtrate it ſynthetically.

RULE.

When a problem has been ſolved algebraically, the demonſtration of it is to be deduced from the ſteps of the algebraic proceſs; by going backward from the end of it to the beginning; obſerving how each ſtep is formed from the foregoing, and forming your proceſs accordingly.

SECT. VIII.

The Reſolution of Equations; and the extraction of their roots in numbers.

PROBLEM LXXXVII.

To find the limits of the roots of an equation.

WHEN an equation is propoſed to have its root extracted, it is proper to find the limits of the roots; leſt we loſe our time in ſeeking the roots beyond theſe limits.

RULE.

Reduce the equation, that the higheſt term may have 1 for its coefficient; then ſquare the coefficient of the ſecond term, from which ſubtract twice the coefficient of the third term, then the ſquare root thereof is greater than the greateſt root of the equation. But the equation ſhould be clear of impoſſible roots.

For that quantity is the ſum of the ſquares of the roots, by Prob. xl. Art. 9. and that ſum, is greater than the ſquare of any one root.

Or thus,

Subſtitute ſeveral numbers ſucceſſively for the unknown quantity; till at laſt you find two numbers which give, one a poſitive, and the other a negative reſult. Then the root is between theſe numbers.

There are other rules among the writers of Algebra, which come nearer; but then they are more laborious.

Ex. 1.

Let $x^3+3x^2-5x-20=0$.

Then $3\times3-2\times-5=9+10=19$.

and $\sqrt{19}=4.3$, *&c.* Therefore 4 3 is greater than any of the roots.

Ex. 2.

Suppoſe $xx-x-5=0$.

If $x=2$, then the reſult is $2-5=-3$.

If $x=3$, the reſult is $6-5=+1$.

Therefore the root is between 2 and -3.

PROBLEM LXXXVIII.

To reſolve a quadratic equation, and extract its root in numbers.

I comprehend all equations under the name of quadratics, in which are two terms involving the unknown quantity; and where the index of one is double to that of the other. As in theſe,

$$aa+ba=d$$
$$a^4+ba^2=d$$
$$a^6+ba^3=d, \text{ \&c.}$$

where b, d, may repreſent any numbers, affirmative or negative.

Every quadratic equation has two roots, though perhaps only one of them will anſwer the queſtion propoſed. And to find theſe roots the equation propoſed muſt be firſt reduced, by dividing all, by the coefficient of the higheſt term; and then tranſpoſing the known quantity to the contrary ſide. Which done, the equation will appear thus, $aa+ba=d$. Now add to both ſides $\frac{1}{4}bb$ the ſquare of half the coefficient of a, and we have $aa+ba+\frac{1}{4}bb=\frac{1}{4}bb+d$, where the firſt ſide is a compleat ſquare; therefore extract the ſquare root, and $a + \frac{1}{2}b = \pm\sqrt{\frac{1}{4}bb + d}$, tranſpoſe $\frac{1}{2}b$, then $a = -\frac{1}{2}b \pm \sqrt{\frac{1}{4}bb + d}$. So a becomes known, being either equal to $-\frac{1}{2}b + \sqrt{\frac{1}{4}bb + d}$, or to $-\frac{1}{2}b - \sqrt{\frac{1}{4}bb + d}$. Whence this

1 RULE.

The equation being cleared, compleat the ſquare by adding to both ſides the ſquare of half the coefficient of the ſecond term. Then extract the root of both ſides, which may be either + or —; then tranſpoſe the known quantity.

Note, If the abſolute number is negative, and greater than $\frac{1}{4}$ the ſquare of the coefficient; the equation is impoſſible.

If $aa + ba = d$,

Then $a = \pm\sqrt{\frac{1}{4}bb + d} - \frac{1}{2}b$.

And the root extracted in numbers gives a; but if $\frac{1}{2}bb$ is leſſer than d, and d negative; it is impoſſible.

Ex. 1.

If $aa + 5a = 68$.

Then $a = \pm\sqrt{68 + 6\frac{1}{4}} - \frac{5}{2} = \pm\sqrt{74.25} - 2.5$;

$$\begin{array}{rl}
 & \dot{7}4.\dot{2}5\ (8.6168\ \&c. \\
 & 64 \\
\hline
166) & 1025 \\
+6) & 996 \\
\hline
1721) & 900 \\
+1) & 1721 \\
\hline
17226) & 117900 \\
+6 & 103356 \\
\hline
17232) & 1454400
\end{array}
\qquad
\begin{array}{rl}
\pm & 8.6168 \\
- & 2.5 \\
\hline
+ & 6.1168 = a \\
\hline
- & 11.1168 = a \\
\hline
\end{array}$$

Ex.

Ex. 2.

Let $aa - 6a = 27$.

Then $a = 3 \pm \sqrt{9 + 27} = 3 \pm \sqrt{36}$.
that is, $a = 3 + 6 = 9$.
or $a = 3 - 6 = -3$.

Ex. 3.

Suppoſe $aa - 236a = -1155$.

Then $a = 118 \pm \sqrt{118^2 - 1155}$;
that is, $a = 118 + 113 = 231$
or $a = 118 - 113 = 5$.

2 RULE.

When you have large numbers to deal with; it is better to proceed thus. Clear the equation;

And if $aa + ba = d$,

then $a = \dfrac{d}{b + a}$, the form.

To find the firſt quotient figure, take $\dfrac{d}{b}$, when b is far greater than a; or take $\sqrt{d}$, when a is far greater than b; or take $\dfrac{d}{2b}$ when a and b are nearly equal; thus it will eaſily be found by a few trials. Or in general, take the firſt figure ſuch, that when it is multiplied by the ſum of itſelf and b, it will produce the firſt figure or figures of d, or the next leſs: this is all the difficulty. Then multiply and ſubtract as uſual, the remainder is the reſolvend.

Then to continue the diviſion; you muſt find a new diviſor for each quotient figure, thus. Add the laſt quotient figure to the laſt diviſor (duly obſerving their places), for a new diviſor; ſee how oft this is contained in the reſolvend, ſet the anſwer in the quotient, and alſo add it to the diviſor; then multiply the whole diviſor by that quotient figure; and ſubtract the product, for a new reſolvend. But when any of the ſigns are negative, the proper quantities are to be ſubtracted, inſtead of being added. This work is always to be repeated for each quotient figure.

When any quotient figure is ſo great that the product exceeds the reſolvend, place a leſs figure in the quotient.

When you have got more than half your intended number of figures in the quotient, you may continue the diviſion without adding the new quotient figures to the diviſor.

Obſerve, each quotient figure is to be added twice to the diviſor; once before multiplication, and once after; juſt as in extracting the ſquare root, and for the ſame reaſon. For this method extracts the ſquare root, when $b = 0$.

When one root is had, the other is found, by adding this to the coefficient b; for the ſum, changing its ſign, is the other root.

This rule is the foundation of the method for extracting the roots of adfected equations.

Ex. 4.

Let $aa + 32a = 4644$.

then $$a = \frac{4644}{32 + a}$$

Suppoſe $\frac{4600}{32} = 100$ too great for a.

$\sqrt{4644} = 69$, which is alſo too great for a. Take $a = \frac{4600}{64} = 7$, too great. Take $a = 50$.

$$\begin{array}{rrl}
32 & & \\
+\ 50 & & \\
\hline
82) & 4644 & (50 \\
+\ 54 & 410\cdot & \\
\hline
136) & 544 & (\ 4 \\
 & 544 & \overline{} \\
 & \overline{} & |\ 54 = a. \\
 & : &
\end{array}$$

Ex. 5.

Let $aa + 35a = 28349994$

$$a = \frac{28349994}{35 + a}$$

Here $a = \sqrt{28}$ &c. $= 5000$ nearly.

$$\begin{array}{rrl}
+35 & & \\
5000 & & \\
\hline
5035) & 28349994 & (5307 = \ddot{a} \\
5300) & 25175\cdots & \\
\hline
10335) & 31749 & \\
307) & 31005 & \\
\hline
10642) & 74494 & \\
 & 74494 & \\
\hline
 & : &
\end{array}$$

Ex. 6.

Suppose $aa - 5307a = - 184520.$

then $a = \dfrac{-184520}{-5307 + a} = \dfrac{184520}{5307 - a}.$

Here $a = \dfrac{184}{5} = 30$ nearly.

5307

5307)	184520 (35 $= a$.
-30	15831·
5277)	26210
-35	26210
5242)	··

Ex. 7.

Let $aa + 463a = 26698$

$$a = \frac{26698}{463+a}.$$

463)	26698 (51.855342 $= a$.
$+50$	2565·
513)	1048
$+51$	564
564)	484.00000
$+1.8$	45264
565.8)	31.3600
$+.85$	283325
566.65)	30275
$+.5$	28335
566.70	1940
· · · ·	1700
	240
	226
	14
	11
	3

Scholium. If $x^4 + bx^2 = d$. Put $a = xx$, then $aa + ba = d$; and find a as above. Then $x = \sqrt{a}$, by extracting the root. And the ſame for higher equations.

To prove the truth of this rule. Let $x+y+z$ &c. be the true value of a; x the firſt figure, y the ſecond, and z the third, *&c.* Then ſince $aa+ba=d$, the value of d will be $b\times\overline{x+y+z}+\overline{x+y+z}|^2$, whence $x+y+z$ &c. or

$$a=\frac{d}{b+a}=\frac{bx+by+bz+\overline{x+y+z}|^2}{b+x+y+z}.$$

The operation.

$$\begin{array}{lll}
 & \left.\begin{array}{l} b \\ +x \end{array}\right) & bx+by+bz+xx+2xy\,(x+y+z\ \&c. \\
 & & \qquad\qquad\quad +yy+2xz \\
 & & \qquad\qquad\quad +zz+2yz \\
\text{1 diviſor} & b+x) & \underline{bx+xx\qquad\qquad\qquad\qquad} \\
 & \underline{+x+y} & \\
\text{2 diviſ.} & b+2x+y) & by+bz+yy+2xy\ \text{reſolvend} \\
 & \underline{+y+z\qquad} & \qquad\ +zz+2xz \\
 & & \qquad\qquad\quad +2yz \\
 & & \underline{by+2xy+yy\qquad\qquad} \\
\text{3 diviſ.} & b+2x+2y+z) & bz+zz+2xz\ \text{reſolvend} \\
 & & \qquad\qquad +2yz \\
 & & \underline{bz+2xz+2yz+zz} \\
 & & \qquad\qquad 0
\end{array}$$

Here $bx+by$ &c. being divided by b gives x in the quotient, and x added to b, gives $b+x$ fo the diviſor, and $bx+xx$ for the product, whic ſubtracted, leaves the reſolvend $by+bz+2x$ $+yy$ &c.

Then in order to get the ſecond figure y, th reſolvend $by+2xy+yy$ &c. is to be divided b $b+2x+y$. Therefore $x+y$ is to be added to th laſt diviſor $b+x$, to get the new diviſor $b+2x+$ This diviſor multiplied by y, gives $by+2xy+y$

whic

which ſubtracted, leaves $bz + 2xz + 2yz + zz$ for the reſolvend.

Then to get the third figure z, it is plain, the reſolvend $bz + 2xz + 2yz + zz$ muſt be divided by the diviſor $b + 2x + 2y + z$, but this new diviſor is $b+2x+y+x+z$, that is, it is the old diviſor with $x + z$ added. Then this diviſor multiplied by z, and ſubtracted, 0 remains. Therefore the root is rightly extracted, and the rule true.

As I am upon this ſubject, I ſhall alſo ſhew the truth of the rule for extracting the ſquare root in Arithmetic, which is the caſe here, when $b = 0$.

Let $\overline{x+y+z}|^2$ be the ſquare, that is,

$$\begin{array}{ll}
1\ \text{div. } x) & xx+2xy+yy+2xz+2yz+zz\ (x+y+z \\
\quad +x) & xx \\
\hline
\quad 2x\ ) & 2xy+yy+2xz+2yz+zz \\
+y & 2xy+yy \\
\hline
2\ \text{div. } 2x+y) & +2xz+2yz+zz \\
\quad +y+z & +2xz+2yz+zz \\
\hline
3\ \text{div. } 2x+2y+z) & 0
\end{array}$$

Here x being the root of the firſt term, its ſquare ſubtracted, leaves the reſolvend $2xy+yy$ &c. Then to find y, the reſolvend muſt be divided by $2x+y$. That is, to the old diviſor x, add $x+y$ for a new diviſor $2x+y$; this multiplied by y, and ſubtracted, leaves the reſolvend $2xz+2yz+zz$. Again to find z, the reſolvend is to be divided by $2x+2y+z$ that 0 remain; that is, to the old diviſor $2x+y$ add $y+z$, the ſum is the new diviſor $2x+2y+z$, which multiplied by z, is equal to the reſolvend, ſo that 0 remains; and the root is $x+y+z$.

PROBLEM LXXXIX.

To extract the root of a cubic equation.

1 RULE.

Take away the ſecond term of the equation, (by Prob. li.) which then will be in this form,

$$a^3 + ba = d.$$

Then ſubſtitute numbers in either of the following forms, and extract the roots, by which means a will be found.

$$a = \sqrt[3]{\tfrac{1}{2}d + \sqrt{\tfrac{1}{4}dd + \tfrac{1}{27}b^3}} - \frac{\tfrac{1}{3}b}{\sqrt[3]{\tfrac{1}{2}d + \sqrt{\tfrac{1}{4}dd + \tfrac{1}{27}b^3}}}.$$

$$\text{or } a = \sqrt[3]{\tfrac{1}{2}d + \sqrt{\tfrac{1}{4}dd + \tfrac{1}{27}b^3}} + \sqrt[3]{\tfrac{1}{2}d - \sqrt{\tfrac{1}{4}dd + \tfrac{1}{27}b^3}}.$$

Note, When b is negative, and $\tfrac{1}{27}b^3$ greater than $\tfrac{1}{4}dd$, the equation is impoſſible.

Ex. 1.

Let $x^3 - 6x = -9$.

Here $b = -6$, $d = -9$, and $\sqrt{\tfrac{1}{4}dd + \tfrac{1}{27}b^3} = \sqrt{12\tfrac{1}{4}} = 3\tfrac{1}{2}$.

and $\sqrt[3]{-4\tfrac{1}{2} + 3\tfrac{1}{2}} = \sqrt[3]{-1} = -1$. Therefore

$$a = -1 - \frac{-2}{-1} = -1 - 2 = -3.$$

or $\sqrt[3]{-4\tfrac{1}{2} - 3\tfrac{1}{2}} = \sqrt[3]{-8} = -2$, whence $a = -1 - 2 \quad -3$, as before.

Ex. 2.

Let $a^3 + 6a = 20$.

Here $b = 6$, $d = 20$, and $\sqrt{\frac{1}{4}dd + \frac{1}{27}b^3} = \sqrt{108}$.
And $\sqrt[3]{10 + \sqrt{108}} = 1 + \sqrt{3}$ (Prob. lxxii.) and $\sqrt[3]{10 - \sqrt{108}} = 1 - \sqrt{3}$. Whence
$a = 1 + \sqrt{3} + 1 - \sqrt{3} = 2$.

Ex. 3.

Let $a^3 - 15a = 4$.

Here $b = -15$, $d = 4$, and $\sqrt{\frac{1}{4}dd + \frac{1}{27}b^3} = \sqrt{-121} = 11\sqrt{-1}$.
And $\sqrt[3]{2 + 11\sqrt{-1}} = 2 + \sqrt{-1}$. And $\sqrt[3]{2 - 11\sqrt{-1}} = 2 - \sqrt{-1}$.
Whence $a = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$.

Ex. 4.

Suppoſe $a^3 + 24a = 587914$.

Here $b = 24$, $d = 587194$. $\sqrt{\frac{1}{4}dd + \frac{1}{27}b^3} = 293957.000878$.
And $\sqrt[3]{\frac{1}{2}d + 29 \text{ \&c.}} = 83.7731$. And $\frac{8}{83.77} = .0958$; therefore
$a = 83.7731 - .0958 = 83.6773$.

SCHOLIUM.

It ſometimes happens that the root may be found, though the negative quantity $\frac{1}{27}b^3$ be greater than $\frac{1}{4}dd$; and that is when the ſurd cubic root can be extracted. For then the irrational parts, in different parts of the equation, will deſtroy one another, and vaniſh; as in Ex. 3.

To prove the truth of this rule. Put $r=\sqrt{\frac{1}{4}dd+\frac{1}{27}b^3}$, $s=\sqrt[3]{\frac{1}{2}d+r}$. Then $a=s-\frac{b}{3s}$, and $a^3=s^3-bs+\frac{bb}{3s}-\frac{b^3}{27s^3}$, and $ba=bs-\frac{bb}{3s}$, therefore $a^3+ba=s^3-\frac{b^3}{27s^3}=\frac{1}{2}d+r-\frac{\frac{1}{27}b^3}{\frac{1}{2}d+r}$ $=\frac{\frac{1}{4}dd+dr+rr-\frac{1}{27}b^3}{\frac{1}{2}d+r}=$ (reſtoring rr) $\frac{\frac{1}{4}dd+dr+\frac{1}{4}dd+\frac{1}{27}b^3-\frac{1}{27}b^3}{\frac{1}{2}d+r}=\frac{\frac{1}{2}dd+dr}{\frac{1}{2}d+r}=d$; that is, $a^3+ba=d$, according to the firſt part of the rule.

And the ſecond part is proved, by ſhewing that $-\frac{b}{3s}=\sqrt[3]{\frac{1}{2}d-r}$. It is plain $\overline{\frac{1}{2}d+r}\times\overline{\frac{1}{2}d-r}$ $=\frac{1}{4}dd-rr=\frac{-1}{27}b^3$, therefore $\frac{1}{2}d-r=\frac{-\frac{1}{27}b^3}{\frac{1}{2}d+r}=$ $-\frac{b^3}{27s^3}$, and $\sqrt[3]{\frac{1}{2}d-r}=-\frac{b}{3s}$. Which was to be proved.

Some of the caſes of cubic equations may alſo be reſolved trigonometrically by the table of ſines. As ſuppoſe the equation $x^3-px=\pm q$, to be given. By Prop. 24, 25. Trigonometry, if $r=$ radius, $y=$ ſine of an arch; then $3y-\frac{4y^3}{rr}=$ S. of thrice the arch. And by Prop. 26. if $x=$ coſine of an arch, then $\frac{4x^3}{rr}-\frac{3}{4}x=$ coſine of thrice that arch.

Theſe equations reduced give $y^3-\frac{3}{4}rry=-\frac{rr}{4}$ $\times$ ſine of thrice the arch. And $x^3-\frac{3}{4}rrx=+\frac{1}{4}rr\times$ coſine of thrice the arch. Or putting y for either the ſine or coſine of the arch, C for the ſine or coſine of thrice the arch; then

$y^3 - \frac{3}{4}rry = \pm\frac{rr}{4}C$, the ſign + being for coſines, and — for ſines.

Then, if the given equation $x^3 - px = \pm q$ is to be reſolved; it muſt be compared with the foregoing, and all the parts made ſimilar in both. Therefore let the equation $x^3 - px = \pm q$, be denoted thus, $x^3 - \frac{3}{4}RRx = \pm\frac{1}{4}RRS$, S being the ſine or coſine of thrice the arch. Therefore $\frac{3}{4}RR = p$, and $R = \sqrt{\frac{4}{3}p}$. Alſo $q = \frac{1}{4}RRS = \frac{1}{3}pS$, and $S = \frac{3q}{p}$. Whence by proportion R $(\sqrt{\frac{4}{3}p})$: S $\left(\frac{3q}{p}\right)$ :: $r : C = \frac{3rq}{p\sqrt{\frac{4}{3}p}}$, the coſine or ſine of an arch. Of which, y is the coſine or ſine of the third part.

Then y being found, it will be $r : y ::$ R $(\sqrt{\frac{4}{3}p})$ $: x = \frac{y\sqrt{\frac{4}{3}p}}{r}$, as required. Hence this

2 RULE.

Take away the ſecond term (by Prob. li.) if it have any; and the equation will be reduced to this form,

$$x^3 - px = \pm q.$$

Then take $\frac{3rq}{p\sqrt{\frac{4}{3}p}}$ = the coſine of an arch (if it be $+q$), or the ſine (if $-q$). Find y = coſine or ſine of $\frac{1}{3}$ that arch; then $\frac{y\sqrt{\frac{4}{3}p}}{r} = x$ required.

And this laſt arch may be either that we found, or, that + 120°, or the ſame + 240°. By which means you will have three roots or values of y.

But note, when $\frac{3rq}{p\sqrt{\frac{4}{3}p}}$ is greater than 1, the queſtion is impoſſible by this rule.

There-

Therefore this rule ſupplies the defect of the firſt rule, which only ſolves equations that have but one root real, and two impoſſible ones: while this rule ſolves ſuch as have three roots real.

Ex. 5.

Let $a^3 - 91a = -330$.

Here $x = a$, $p = 91$, $q = 330$, and $\sqrt{\frac{4}{3}p} = 11.015$, and if $r = 1$, $\frac{3rq}{p\sqrt{\frac{4}{3}p}} = .987655$ = ſine of 81° very near; and the third part is 27, or 147, or 267; whoſe ſines are, $y = .45399$, or $.54467$, or $-.99863$; theſe multiplied by 11.015 produce 5.0004, and 5.9991, and -10.9998; therefore the three roots are 5, 6, and -11.

Ex. 6.

Suppoſe $x^3 - 19x = 30$.

Here $p = 19$, $q = 30$, and $\sqrt{\frac{4}{3}p} = 5.03323$; and $\frac{3rq}{p\sqrt{\frac{4}{3}p}} = .94112$ = coſine of 19° 45′½, and the third part is 6° 35′, or 126° 35′, or 246° 35′, whoſe coſine is $y = .99340$, or $-.59599$, or $-.39741$; which multiplied by 5.03323, produce 4.99998, and -2.99974, and -2.00024. So the three roots are theſe; 5, -3, and -2.

PROBLEM XC.

To reſolve a biquadratic equation, by diſſolving it into two quadratics.

Take away the ſecond term (by Prob. li.), and let the reſulting equation be $x^4 + qx^2 + rx + s = 0$. Suppoſe it to be generated by the two quadratics, $xx + ex + f = 0$, and $xx - ex + g = 0$. Theſe being multiplied together produce

$$\begin{array}{lllll} x^4 & +fx^2 & +egx & +fg & =0. \\ & +g & -ef & & \\ & -ee & & & \end{array}$$

Comparing the terms of this with the firſt equation, we have $f+g-ee=q$, $eg-ef=r$, and $fg=s$; whence $g+f=q+ee$, and $g-f=\frac{r}{e}$; and conſequently $g=\frac{q+ee+\frac{r}{e}}{2}$, and $f=\frac{q+ee-\frac{r}{e}}{2}$. And $(fg=)\ \frac{qq+2qee+e^4-\frac{rr}{ee}}{4}=s$. And by reduction,

$$\begin{array}{lllll} e^6 & +2qe^4 & +qqee & -rr & =0. \\ & & -4s & & \end{array}$$

Put $y=ee$, and then

$$\begin{array}{lllll} y^3 & +2qy^2 & +qqy & -rr & =0. \\ & & -4s & & \end{array}$$

A cubic equation; whence the following

RULE.

To reſolve the biquadratic equation $x^4+qx^2+rx+s=0$. Take the cubic equation

$$\begin{array}{lllll} y^3 & +2qyy & +qqy & -rr & =0; \\ & & -4s & & \end{array}$$

out of which take away the ſecond term (by Prob. li.); and find the root by the laſt problem, or otherwiſe; and from thence find y. Then take $e=\sqrt{y}$, and $f=\frac{q+ee-\frac{r}{e}}{2}$, and $g=\frac{q+ee+\frac{r}{e}}{2}$.

Laſtly, find the roots of theſe two quadratic equations, $xx+ex+f=0$, and $xx-ex+g=0$. And theſe will be the four roots, of the biquadratic $x^4+qx^2+rx+s=0$.

Example.

Let $x^4 - 25x^2 + 60x - 36 = 0$.

From this you have the cubic equation $y^3 - 50y^2 + 769y - 3600 = 0$. Take away the ſecond term, by writing $v + \frac{50}{3}$ for y. And we have $v^3 - \frac{193}{3}v = \frac{1150}{27}$. And by Rule 2. Prob. laſt, $v = 8.3333$ &c. $= 8\frac{1}{3}$, whence $y = 8\frac{1}{3} + \frac{50}{3} = 25$, and $e = 5$; therefore $f = \frac{-25+25-\frac{60}{5}}{2} = -6$, and $g = \frac{-25+25+\frac{60}{5}}{2} = +6$. Whence $xx + 5x - 6 = 0$, and $xx - 5x + 6 = 0$; and the roots of the former equation are 1 and -6; and of the latter, 3 and 2. Therefore the four roots of the biquadratic, $x^4 - 25x^2 + 60x - 36 = 0$, are 1, 2, 3, and -6.

And the ſame roots will be found, by making uſe of the other values of v, which are $-\frac{2}{3}$, and $-\frac{23}{3}$.

Schol. But this and ſuch like rules are of little value; for there is far more labour here in getting the roots than by the method of converging ſeries, which is to follow.

PROBLEM XCI.

To extract the root of any pure power in numbers.

Let G be the number given to be extracted; m the root required, r the neareſt root, and e

the remaining part of it; then $\overline{r+e}|^m = G$, that is (Cor. 1. Prob. v.) $r^m + mr^{m-1}e + m.\frac{m-1}{2}r^{m-2}ee + m.\frac{m-1}{2}.\frac{m-2}{3}r^{m-3}e^3$ &c. $= G$, and rejecting e^3 and the higher powers, as very ſmall; we have $mr^{m-1}e + m.\frac{m-1}{2}r^{m-2}ee = G - r^m$, and

$$\frac{mr^{m-1}}{m.\frac{m-1}{2}r^{m-2}}e + ee = \frac{G-r^m}{m.\frac{m-1}{2}r^{m-2}}.$$

Hence this

1 RULE.

Let G = abſolute number.
r = the neareſt root you can find.
$r+e$ = the true root.
m = the index of the root.

$$b = \frac{2r}{m-1}.$$

$$D = \frac{G-r^m}{m.\frac{m-1}{2}r^{m-2}}$$

Then $be + ee = D$, or $e = \frac{D}{b+e}$ nearly.

Which equation is to be reſolved by Prob. lxxxviii. When e is had, then $r+e$ is to be taken for a new value of r, and the operation repeated, perhaps oftener than once. This rule generally triples the number of figures.

But if the third power of e be taken in, then $mr^{m-1}e + m.\frac{m-1}{2}r^{m-2}ee + m.\frac{m-1}{2}.\frac{m-2}{3}r^{m-3}e^3 = G - r^m$ and ſince $be + ee = D$, therefore $ee = D - be$, and $e^3 = De - bee$, and $m . \frac{m-1}{2} . \frac{m-2}{3} r^{m-3} e^3 = m . \frac{m-1}{2} . \frac{m-2}{3} r^{m-3} \times \overline{De - bee}$, whence $mr^{m-1} e + m . \frac{m-1}{2} r^{m-2} ee + m . \frac{m-1}{2} . \frac{m-2}{3} r^{m-3} \times \overline{De - bee} = G - r^m$, and $rre + \frac{m-1}{2} . \frac{m-2}{3} De + \frac{m-1}{2} ree - \frac{m-1}{2} . \frac{m-2}{3} bee = \frac{G - r^m}{mr^{m-3}} = rF$, by ſubſtitution, (putting $F = \frac{G - r^m}{mr^{m-2}}$); expunge b and D, then $rre + \frac{m-1}{2} . \frac{m-2}{3} \times \frac{G - r^m}{m . \frac{m-1}{2} r^{m-2}} e + \frac{m-1}{2} ree - \frac{m-1}{2} . \frac{m-2}{3} \times \frac{2r}{m-1} ee = rF$; that is, $rre + \frac{m-2}{3} \times Fe + \frac{m-1}{2} ree - \frac{m-2}{3} ree = rF$, that is, $rr + \frac{m-2}{3} F \times e + \frac{m+1}{6} ree = rF$, whence this

2 RULE.

Let G = abſolute number.
r = neareſt root you can find.
$r + e$ = true root.
m = index of the root.
$F = \frac{G - r^m}{mr^{m-2}}$.

Then $\frac{6r + \frac{2m-4}{r} . F}{m+1} e + ee = \frac{6F}{m+1}$, nearly.

Which is to be ſolved as Prob. lxxxviii, and repeated with new r, if there be occaſion. This rule commonly quintuples the number of figures in the root, true; at each operation.

The root of any number may alſo be extracted by Prob. lviii. after this manner.

3 RULE.

Let P+Pq, be the number given to be extracted.
P, the greateſt power contained in it.
Pq, the remainder; and
q, the quotient ariſing by dividing the remainder by the greateſt power.
n, the index of the root. Then

$$\sqrt[n]{P+Pq} = P^{\frac{1}{n}} + \frac{1}{n}Aq - \frac{n-1}{2n}Bq - \frac{2n-1}{3n}Cq - \frac{3n-1}{4n}Dq \text{ \&c.}$$

Where A, B, C, &c. are the preceding terms. In this rule, when two or three figures are got, put them equal to $P^{\frac{1}{n}}$, and begin the operation anew; and the ſeries will then converge exceeding faſt; and ſo much faſter as q is leſs.

Cor. *Hence it follows, that*

$$\sqrt[2]{P+Pq} = \sqrt{P} + \frac{1}{2}Aq - \frac{1}{4}Bq - \frac{3}{6}Cq - \frac{5}{8}Dq - \frac{7}{10}Eq$$ &c. for the ſquare root.

$$\sqrt[3]{P+Pq} = \sqrt[3]{P} + \frac{1}{3}Aq - \frac{2}{6}Bq - \frac{5}{9}Cq - \frac{8}{12}Dq - \frac{11}{15}Eq$$ &c. for the cube root.

$$\sqrt[4]{P+Pq} = \sqrt[4]{P} + \frac{1}{4}Aq - \frac{3}{8}Bq - \frac{7}{12}Cq - \frac{11}{16}Dq$$ &c. for the biquadrate root.

$\sqrt[5]{\ }$

$$\sqrt[5]{P+Pq} = \sqrt[5]{P} + \frac{1}{5}Aq - \frac{4}{10}Bq - \frac{9}{15}Cq - \frac{14}{20}Dq \text{ \&c for the fifth root.}$$

$$\sqrt[3]{\overline{P+Pq}|^{2}} = P^{\frac{2}{3}} + \frac{2}{3}Aq - \frac{1}{6}Bq - \frac{4}{9}Cq - \frac{7}{12}Dq - \text{\&c. for the cube root of the ſquare; and ſo on.}$$

Ex. 1.

What is the cube root of 2?

Here $G = 2$, $r = 1$, $m = 3$, by Rule 1, $b = 1$; $D = \frac{1}{3} = .3333$; and $e = \frac{D}{b+e}$.

$$\begin{array}{ll}
1. & \\
+\ 2 \big) & .3333\ (.26 = e \\
 & 24 \\
\hline
1.2\ \ \big) & .0933 \\
+\ 26 & \ \ 876 \\
\hline
1.46 & \ \ \ 57
\end{array}
\qquad \text{and } r + e = 1.26$$

Again, for a ſecond operation;

Let new $r = 1.26$; then $G - r^3 = -.000376$, and $m.\frac{m-1}{2}r = 3r = 3.78$, and $D = \frac{.000376}{3.78} = .000099471$, and becauſe e is negative here, $e = \frac{.000099471}{1.26 - e}$.

$$\begin{array}{rr}
1.2600000) & .0000994710\,(-.000078950106 \\
-7 & 881951 \\ \hline
1.2599300 & 11275900 \\
-78 & 10078816 \\ \hline
1.2598520 & 11970840 \\
-89 & 11338587 \\ \hline
1.259843 & 632253 \\
\cdot\cdot\cdot\cdot\cdot & 629921 \\ \cline{2-2}
 & 2332 \\
 & 1259 \\ \cline{2-2}
 & 73
\end{array}$$

$$\begin{array}{rl}
r = & 1\,.\,26000 \\
e = & -.000078950106 \\ \hline
\sqrt[3]{2} = & 1\,.\,259921049894
\end{array}$$

Ex. 2.

Extract the 5th root of 2327834559873.

Firſt point every fifth figure thus

$$23\dot{2}7834\dot{5}5987\dot{3},$$

Then for brevity's ſake, take only the firſt period, as an integer, that is 232. Then proceeding by Rule 2, we ſhall find 2 the root of the greateſt power contained therein; and thence, $r = 2$, and

$$\begin{array}{rrl}
 & 232 & = G \\
 & 32 & = r^m \\ \cline{2-3}
 & 200 & = G - r^m \\
\text{and} & 40 & = mr^{m-2} \\
\text{Whence} & \dfrac{200}{40} & = 5 = F.
\end{array}$$

Therefore $4\frac{1}{2}e + ee = 5$, or $e = \frac{5}{4.5 + e}$.

$$\begin{array}{l l}
4.5) & 5.00\ (.92 = e. \\
+9 & 4.86 \\
\hline
5.4\ ) & 1400 \\
+92 & 1264 \\
\hline
6.32) & 136
\end{array}$$

whence $r = 2.92$;

Suppoſe again new $r = 290$.

Then $r^3 = 24389000$.

$$\begin{array}{r c l}
G & = & 2327834559873 \\
r^5 & = & 2051114900000 \\
\hline
G - r^5 & = & 276719659873 \\
F & = & 2269.217
\end{array}$$

whence

$$297.825e + ee = 2269.217$$

$$\text{or } e = \frac{2269.217}{297.825 + e}$$

$$\begin{array}{l l}
297.825) & 2269.217\ (7.43375 \\
+7 & 2133.775 \\
\hline
304.825) & 1354420 \\
+7.4 & 1248900 \\
\hline
312.225) & 105520 \\
+43 & 93796 \\
\hline
312.65|5 & 11724 \\
+\ 33 & 9380 \\
\hline
312.68 & 2344 \\
:\ \cdot\cdot\!: & 2188 \\
\hline
 & 156
\end{array}$$

Whence $r + e = 297.4337$, which may be taken for new r, and the operation repeated, if there be occaſion.

Ex.

Ex. 3.

What is the 7th root of 100000.

The neareſt root of 100000 is 5, whence by Rule 3d,

$$\begin{array}{rl} P + Pq = & 100000 \\ P = & \underline{78125} \\ Pq = & 21875. \\ \text{whence } q = & .28 \text{ \&c.} \end{array}$$

And $\sqrt[7]{P+Pq} = \sqrt[7]{P} + \frac{1}{7}Aq - \frac{6}{14}Bq - \frac{13}{21}Cq - \frac{20}{28}Dq - \frac{27}{35}Eq$ &c.

That is,

$$\begin{array}{rl} \sqrt[7]{P} = & +5.000 \\ +\frac{1}{7}Aq = & .200 \\ -\frac{3}{7}Bq = & -024 \\ -\frac{13}{21}Cq = & +.004 \\ -\frac{5}{7}Dq = & -001 \\ \text{\&c.} & \overline{} \end{array}$$

$$5.204 - .025 = 5.179$$

But becauſe this converges ſlow, take 5.179 for the root, and involve it to the 7th power, and we have

$$\begin{array}{rl} P + Pq = & 100000. \\ P = & \underline{99935.8652873094} \\ Pq = & 64.1347126906 \\ q = & .0006417587 \end{array}$$

$$\begin{array}{rll} \text{Then } \sqrt[7]{P} = & 5.1790000000 & = A \\ +\frac{1}{7}Aq = & +.0004748097 & = B \\ -\frac{3}{7}Bq = & - \qquad 1306 & = C \\ -\frac{13}{21}Cq = & + \qquad\quad 1 & = D \\ & \overline{5.1794748098} & \\ & -1306 & \\ & \overline{5.1794746792} & = \sqrt{100000}. \end{array}$$

Schol. 1. If the root is required for only a few places of figures; the eaſieſt way by far, is to extract it by the help of logarithms.

Schol. 2. From the foregoing proceſs, the rule for extracting the cube root in arithmetic, may be demonſtrated.

Let $a + e$ be the root, a the firſt figure, e the ſecond. Then the cube is $a^3 + 3a^2e + 3ae^2 + e^3$; then a^3 the greateſt cube contained in it, being ſubtracted; there remains $3a^2e + 3ae^2$, ſetting aſide e^3 as being very ſmall. Divide this remainder by 3, and we have $a^2e + aee$, from which to find e, this remainder or reſolvend muſt be divided by $aa + ae$. That is, the reſolvend muſt be divided by aa, the ſquare of the root, and then to the diviſor, there muſt be added ae, the product of the root by the quotient figure; and the whole will be the true diviſor for finding e. But as e^3 was left out of the account; the root got this way will deviate from the true root; and therefore you muſt, after a few figures are had, begin the operation again, with the new root which you have already got.

PROBLEM XCII.

To extract the root of any adfected equation, in numbers.

Preparation.

Suppoſe $Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5$ &c. $= N$. Put $r + e = x$, r being the firſt figure of the root; and to find r, put 1, 10, 100 ſucceſſively for x; and the neareſt value of theſe being found, try the intermediate numbers 5, 50, *&c.* then expunging x, we have

$Ar + Ae$

$$\left.\begin{array}{ll} Ar + Ae & = Ax \\ Br^2 + 2Bre + Bee & = Bxx \\ Cr^3 + 3Cr^2e + 3Cree + Ce^3 & = Cx^3 \\ Dr^4 + 4Dr^3e + 6Dr^2e^2 + 4Dre^3 + De^4 & = Dx^4 \\ \&c. & \&c. \end{array}\right\} = N.$$

Sum $P + ae + bee + ce^3 + de^4$ &c. $= N$.

And $ae + bee + ce^3 + de^4$ &c. $= N - P = f$.

Then ſince $e = \frac{f}{a}$ nearly, we ſhall have $ae + be \times \frac{f}{a} = f$. Or $ae + \frac{bf}{a} e = f$. From whence we ſhall have this

1 RULE.

$$e = \frac{f}{a + b \times \frac{f}{a}} \text{ nearly.}$$

Or, if more exactneſs be required, we may bring in ee; then $ae + bee = f$, whence this

2 RULE.

$$\frac{a}{b} e + ee = \frac{f}{b}, \text{ or } e = \frac{\frac{f}{b}}{\frac{a}{b} + e} - \frac{ce^3 + de^4 \ \&c.}{a + be}$$

nearly, to be wrought by LXXXVIII. Rule 2.

Or if e^3 be taken in for more exactneſs; proceed thus, $bee = f - ae$, and $ce^3 = \frac{cfe - caee}{b}$, whence $ae + bee + ce^3 = ae + bee + \frac{cf}{b} e - \frac{ca}{b} ee = f$, whence

3 RULE.

$\dfrac{a + \dfrac{cf}{b}}{b - \dfrac{ca}{b}} e + ec = \dfrac{f}{b - \dfrac{ca}{b}}$, very near; to be wrought as Prob. lxxxviii. Rule 2.

In any of theſe rules the operation muſt be repeated after a few figures are had, by taking a new value of r, and proceeding as before.

Ex. 1.

Let $120x^3 + 3657x^2 - 38059x = 8007115$.

By a few trials, you will find x to be greater than 30, and leſs than 40. Therefore ſuppoſe $r = 30$, and $30 + e = x$ the root ſought, which being involved, and taking the leaſt powers firſt, as in the rule, we have

$$\left.\begin{array}{l} -1141770 - 38059e \\ +3291300 + 219420e + 3657ee \\ +3240000 + 324000e + 10800ee + 120e^3 \end{array}\right\} = 8007115.$$

Which being added,

$$\begin{array}{lrl} & 5389530 + 505361e + 14457ee + 120e^3 & = 8007115 \\ \text{and} & 505361e + 14457ee + 120e^3 & = 2617585. \\ \text{or} & ae + bee + ce^3 & = f. \end{array}$$

Then to ſhorten the work, divide by 1000, and then $505e + 14ee$ &c. $= 2617$, and by Rule 1, $e = \dfrac{2617}{505} = 5.18$; or rather $e = \dfrac{2617}{505 + 14 \times 5.18} = \dfrac{2617}{577} = 4.53$. Whence $r + e$ or $x = 34.5$ for a new operation. Which being involved, beginning at the higheſt power firſt, we have

$$\left.\begin{array}{l} \,4927635 + 42849e + 12420ee \\ + 4352744\tfrac{1}{4} + 252333e + 3657ee \\ - 1313035\tfrac{1}{2} - 38059e \end{array}\right\} = 8007115.$$

That is,

$$7967343\tfrac{3}{4} + 257123e + 16077ee = 8007115.$$

or $$257123e + 16077ee = 39771.25$$

whence $$15.9932e + ee = 2.473799$$

and by rule 2d, $e = \frac{2.473799}{15.9932 + e}$.

$$\begin{array}{ll}
15.9932) & 2.473799\ (.1532 = e \\
+.1 & 1.60932 \\ \hline
16.0932) & 864479 \\
+15 & 812160 \\ \hline
16.2432) & 52319 \\
+.53 & 48888 \\ \hline
16\ 296|2) & 3431 \\
+3 & 3260 \\ \hline
16.30 & 171
\end{array}$$

$$\begin{array}{l}
r = 34.5 \\
e = .1532 \\ \hline
r + e = 34.6532 = x.
\end{array}$$

Ex. 2.

Let $z^4 - 3z^2 - 75z = 10000.$

Here by a few trials z will be found very near 10. Therefore let $r = 10$, and $r + e = z$. Then

$$\left.\begin{array}{l}
z^4 = 10000 + 4000e + 600ee \text{ \&c.} \\
-3z^2 = -300 - 60e - 3ee \\
+75z = +750 + 75e
\end{array}\right\} = 10000.$$

Being added,

$$10450 + 4015e + 597ee = 10000.$$

or $$4015e + 597ee = -450$$

or $$6.725e + ee = -0.753769$$

Therefore e is negative, and by Rule 2.

$$e = \frac{-0.753769}{6.725 + e}.$$

$$\begin{array}{lll}
6.725) & -0.753769\ (-0.114 = e & \\
-1 & -6625 & \\
\hline
6.625) & -9126 & r = 10.000 \\
-11 & -6515 & e = -.114 \\
\hline
6.515) & -26119 & r + e = 9.886 = z. \\
-14 & -26004 & \\
\hline
6.501 & -115 & \\
\hline
\end{array}$$

Again, put $r = 9.886$, and $r + e = z$; then

$$\begin{array}{ll}
9551.738507135 & + 3864.753593824e \\
-293.198988 & - 59.316e \\
+741.450 & + 75e \\
+586.397976ee & = 10000 \\
-3ee &
\end{array}$$

; that is by addition, $9999.989519135 + 3880.437593824e + 583.397976ee = 10000$; and tranſpoſing,

$3880.437593824e + 583.397976ee = .010480865.$

And $e = \dfrac{.010480865}{3880.437} = .00000270095$, nearly.

Then $r + e = 9.88600270095 = z.$

Ex. 3.

Suppoſe $7y^5 + 2100y^3 - 8000y^2 = 385000000.$

By a few trials y will be found between 50 and 60; therefore put $r = 50$, and $r + e = y$; then expunge y. Or rather thus: Since the numbers are large, transform the equation (by Prob. xlii.), by putting $x = \frac{1}{10}y$, or $y = 10x$, which done we have $700000x^5 + 2100000x^3 - 800000x^2 = 385000000$, or $7x^5 + 21x^3 - 8x^2 = 38500$. Then to extract the root of this, put $r = 5$, and $r + e$ or $5 + e = x$; and x being expunged, we have

21875

$$\left.\begin{array}{rrrr} 21875 + & 21875e + & 8750ee + & 1750e^3 \\ +2625 + & 1575e + & 315ee + & 21e^3 \\ -\ 200 - & 80e - & 8ee & \end{array}\right\} = 38500.$$

That is,

$$24300 + 23370e + 9057ee + 1729e^3 = 38500.$$

$$\text{and} \quad 23370e + 9057ee + 1729e^3 = 14200.$$

Then by Rule 3, $\frac{cf}{b} = 2711$, $\frac{ca}{b} = 4461$, whence

$$5.675e + ee = 3.0900.$$

$$\begin{array}{ll} 5.675)\ & 3.0900\ (.5004 = e \\ +\ 5 & 30875 \\ \hline 6.175 & \quad 25 \\ +\ 5 & \\ \hline 6.675 & \end{array}$$

Again, put $r = 5.5$, and repeating the operation,

$$\left.\begin{array}{rrr} 35229.90625 + & 32027.187e + & 11646.25ee \\ +3493.875 + & 1905.75e + & 346.5ee \\ -\ 242. - & 88.e & -8.ee \end{array}\right\} = 38500.$$

That is when added,

$38481.78125 + 33844.937e + 11984.75ee = 38500.$ And

$$33844.937e + 11984.75ee = 18.21875.$$

Then $\frac{18.21875}{33845} = .0005383 = \frac{f}{a} = e$ nearly.

Then $b \times \frac{f}{a} = 6{,}451$, and (Rule 1.)

$\frac{18.21875}{33844.937 + 6.451} = .000538198 = e$, more exactly. Then

$r + e = 5.500538198 = x$, and $y = 10x = 55.00538198$.

The root may alſo be extracted as follows. Having got $ae + be^2 + ce^3 + de^4 = f$, as before directed; let v be the firſt figure of the value of e, s the ſecond. Then putting $v + s$ for e; $a \times \overline{v + s} + b \times \overline{v + s}|^2 + c \times \overline{v + s}|^3$ &c. $= f$; that is, $av + as + bv^2 + 2bvs$ &c. $+ cv^3 + 3cv^2s$ &c. $= f$. And $as + 2bvs + 3cv^2s$ &c. $= f - av - bv^2 - cv^3$ &c. Whence

$$s = \frac{f - av - bv^2 - cv^3 \text{ \&c.}}{a + 2bv + 3cv^2 \text{ \&c.}}$$ Whence this

4 RULE.

Having any equation given, proceed as in the other rules, till you get $ae + bee + ce^3 + de^4$ &c. $= f$. Then find by repeated trials, the firſt figure v, of the value of e, ſo that $v \times \overline{a + bv + cv^2 + dv^3}$ &c. may be nearly $= f$; and take that product from f to find the remainder.

Then to find the next figure or figures; divide this remainder by $a + bv + 3cvv + 4dv^3$ &c. the quotient is the ſaid figure, which muſt be added to v, for a new value of v. Then repeat the operation with new v, *viz.* take $v \times \overline{a + bv + cv^2 + dv^3}$ &c. from f, and divide the remainder by $a + 2bv + 3cv^2$ &c. and add the quotient to laſt v; and ſo on.

And note, after the diviſor once takes place, each new quotient may be continued to near as many figures, as all the preceding ones. Alſo in the diviſor, you need not continue the parts of the diviſor $2bv$, $3bv^2$ &c. any farther in decimals, than to anſwer the number of figures, you would have true in the root.

General form.

$v = \frac{f}{a}$; or $v = \frac{f}{a + bv}$ nearer; or $v = \frac{f}{a + bv + cvv}$, nearer ſtill; *&c.* then, next figure $= \frac{f - av - bv^2 - cv^3 \text{ \&c.}}{a + 2bv + 3cvv \text{ \&c.}}$.

Ex. 4.

Let $z^3 - 17z^2 + 54z = 350$.

Here z is greater than 10, and leſs than 20. Let $r = 10$, $r + e = z$; then

$$\left.\begin{array}{rrrr} 1000 & +\ 300e & +\ 30e^2 & +\ e^3 \\ -\ 1700 & -\ 340e & -\ 17ee & \\ +\ 540 & +\ 54e & & \end{array}\right\} = 350.$$

$$\begin{array}{lrrrrl} \text{or} & -\ 160 & +\ 14e & +\ 13e^2 & +\ e^3 & = 350. \\ \text{that is} & & 14e & +\ 13ee & +\ e^3 & = 510. \end{array}$$

To find e, try 1, 2, 3, *&c* and you will find e very near 5, but ſomething leſs. Therefore take $v = 5$, and $v \times \overline{a + bv + cv^2} = 5 \times \overline{14 + 65 + 25} = 520$, and $510 - 520 = -10$, then $a + 2bv + 3cv^2 = 219$, and

$$\frac{-10}{219} = -.045 \qquad \begin{array}{r} v = 5.000 \\ -\ .045 \\ \hline e = 4.955 \end{array}$$

Let new $v = 4.95$; then $\overline{a + bv + cv^2} \times v = 509.119875$, and $510 - 509.119875 = 0.880125$. Alſo $a + 2bv + 3cv^2 = 216.2075$. Whence $s = \frac{0.880125}{216.2075} = .00407$, and $e = 4.95407$;

whence $z = 14.95407$. Or, if you pleaſe, put $v = 4.95407$ for a new operation.

Ex. 5.

Let $2x^4 - 16x^3 + 40x^2 - 30x = -1$.

By a few trials, it appears that x is between 1 and 2. Therefore put $r = 1$, $r + e = x$. Then expunging x,

$$\left.\begin{array}{rrrrr} 2 & +\ 8e & +\ 12ee & +\ 8e^3 & +\ 2e^4 \\ -\ 16 & -\ 48e & -\ 48ee & -\ 16e^3 & \\ +\ 40 & +\ 80e & +\ 40ee & & \\ -\ 30 & -\ 30e & & & \end{array}\right\} = -1.$$

The ſum is

$$\begin{array}{rrrrrl} -4 & +\ 10e & +\ 4ee & -\ 8e^3 & +\ 2e^4 & = -1 \\ \text{or} & 10e & +\ 4ee & -\ 8e^3 & +\ 2e^4 & = 3 \\ & a & b & c & d & \ \ f \end{array}$$

Here we have $e = \frac{3}{10} = .3$, or more exactly $e = \frac{3}{10+4e} = \frac{3}{11.2} = .26 = v$.

Then for the next figures of the root, $v\times\overline{a+bv+cv^2+dv^3} = 2.73893$, and $3 - 2.73893 = .26106$. Alſo $a + 2bv + 3cv^2 + 4dv^3 = 10.598$, and

$$\begin{array}{r} \frac{.26106}{10.598} = .0246 \\ v = .26 \\ \hline \text{then}\quad e = .2846 \\ \hline \end{array}$$

Take new $v = .2846$, then $v\times\overline{a+bv+cv^2+dv^3} = 2.99869539$, and $3 - 2.99869539 = .00130461$. Alſo $a + 2bv + 3cv^2 + 4dv^3 = 10.51728$. Then

.00130461

$$\frac{.00130461}{10.5173} = .00012404$$

$$\begin{array}{l} .2846 = v \\ \hline .28472404 = e, \end{array}$$

whence $r + e$ or $x = 1.28472404$.

The roots of equations may alſo be extracted by help of the Rule of Falſe Poſition in Arithmetic, as follows.

5 RULE.

In ſuch equations as contain ſurds, exponential quantities, *&c.* make two ſuppoſitions in numbers, for the root, as near as you can get them. Then each of theſe being put in the equation inſtead of the root, you muſt mark the errors (that is, the exceſs or defect) ariſing from each of them.

Then multiply the difference of the ſuppoſed numbers by the leaſt error, and divide the product, by the difference of the errors, if they are like, (that is, both exceſſes or both defects); or by the ſum, if unlike. Then

The quotient is the correction of the number belonging to the leaſt error; and is to be added if that number was too little; or ſubtracted, if too great. This gives the root nearer than before.

In like manner try this root, and the neareſt of the former, or elſe take a new ſuppoſed number; then find their errors, and proceed as before, and you will get a root ſtill nearer. And thus by repeating the operation, you may continually approximate, as near as you will, to the true root.

Ex.

Ex. 6.

Suppoſe $x^x = 100$, *to find* x.

By the nature of logarithms $x \times \log{:}\, x = \log.\ 100 = 2$.

Here x, by a few trials, will be found greater than 3, and leſs than 4. Suppoſe $x = 3\frac{1}{2}$; then $l{:}x = .5440680$, and $xl{:}x = 1.9042380$, which ſhould be equal to

$$\begin{array}{r} 2 \\ \hline -.0957620 \end{array} = 1 \text{ Er. too little.}$$

Again, ſuppoſe $x = 3.6$, then $l{:}x = .5563025$, and $xl{:}x = 2.0026890$

$$\begin{array}{r} 2 \\ \hline +.0026890 \end{array} = 2 \text{ Er. too great.}$$

Hence we have

1 num.	3.5	1 er.	$-.095762$
2 num.	3.6	2 er.	$+.002689$
diff.	0.1	ſum	.098451

Then $\dfrac{0.1 \times .002689}{.09845} = .00273 = \text{cor.}$

$$\begin{array}{lr} \text{2 num.} & 3.60000 \\ \text{correct.} & -.00273 \\ \hline & 3.59727 = x. \end{array}$$

Again, ſuppoſe $x = 3.597$, then $l{:}x = .5559404$, and $xl{:}x = 1.9997176$, which ſubtracted from 2, gives $-.0002824$ the error, too little. Whence

2 num.	3.600,	2 er. =	$+ .0026890$
3 num.	3.597,	3 er. =	$- .0002824$
diff.	.003,	ſum	.0029714

Then $\frac{.003 \times .0002824}{.0029714} = .000285$ the cor.

$$\begin{array}{lr} 3 \text{ num.} & 3.597 \\ \text{cor.} + & .000285 \\ \hline x = & 3.597285 \end{array}$$ as required.

Ex. 7.

If s be the ſine of an arch z, rad. $=1$, and $4sz=5$, to find s and z.

By diviſion $sz = 1.25$. The length of 1 degree is $= .01745329$ &c. By a few trials, we may find z between 70 and 80 degrees. Suppoſe $z = 70$ deg. then $.01745 \times 70 = 1.2215$; alſo S. 70 $= .939 = s$, and $sz = 1.1469$, and $1.25 - 1.1469 = .1031$ the firſt error, too little.

Again, ſuppoſe $z = 75$ deg. then $.01745 \times 75 = 1.3087$, and $s = .966$, and $sz = 1.2642$; and $1.2642 - 1.25 = 0142$ the ſecond error, too much. Hence

$$\begin{array}{lr} 1 \text{ num.} & 70 \\ 2 \text{ num.} & 75 \\ \hline & 5. \end{array} \qquad \begin{array}{lr} 1 \text{ er.} & -.1031 \\ 2 \text{ er.} & +.0142 \\ \hline \text{ſum} & .1173 \end{array}$$

Then $\frac{5 \times .0142}{.1173} = \frac{.0710}{.117} = .60$ the cor.

$$\begin{array}{lr} 2 \text{ num.} & 75.0 \\ - \text{cor.} & .6 \\ \hline z = & 74.4 \end{array}$$

Again, let $z = 74.4 = 74° : 24°$, $s = .9631626$; then $.01745329 \times 74.4 = 1.298524$, and $1.298524s = 1.2506895$, from which ſubtract 1.25, then $.0006895 =$ 3d error, too much.

2 num.

2 num.	75	2 er.	$+.0143937$
3 num.	74.4	3 er.	$+.0006895$
diff.	.6	diff.	.0133105

Then $\frac{.6\times.0006895}{.01331}=.0310.$

3 num. 74.400
— cor. .031

and $z = 74.369 = 74° : 22' : 8''$
and $s = .9630372.$

SCHOLIUM.

There are alſo other ways of extracting the roots of equations, though not much different from ſome of the foregoing ones, particularly a method of Sir *I. Newton*'s, which is like the proceſs uſed in the ſecond method foregoing; the principal difference being, that he every where takes a new letter, where we find a new value of e.

Alſo ſurd or tranſcendental equations, may be reſolved by reducing ſome of the quantities to infinite ſeries; proceeding by the rules of Sect. VI.

In equations, where the terms involve a great many factors, which makes it tedious to multiply them together; it will be a ſhorter way to add the logarithms of the ſeveral factors together; and then find the number belonging, which will be the numeral coefficient of that term. And thus all the coefficients of the particular terms may be found.

We may note, that though the third rule converges faſter than the reſt; yet, as there is ſo much trouble in finding the coefficients, and diviſors, it will be found not ſo expeditious as the ſecond, or even the firſt.

In making uſe of the ſecond rule, after half the number of places are found for the value of e; it will be needleſs to form new diviſors; for the reſt of the figures will be as truly found by plain diviſion. For what is added to the diviſor, in places ſo far back, does not at all affect the quotient.

The root may alſo be extracted as in the following problem, and the coefficients a, b, c, &c. found as there directed; which is a compendious method, when the equation conſiſts of many terms.

PROBLEM XCIII.

To extract the root of the infinite ſeries $Az + Bz^2 + Cz^3 + Dz^4 + Ez^5$ &c. $= N$, *in numbers; ſuppoſing this ſeries to converge faſt enough.*

Preparation.

Take r as near the root z, as you can find it; and let $r + e = z$, and z being expunged, we have

$$\left.\begin{array}{l} Ar + Ae \\ + Br^2 + 2Bre + Be^2 \\ + Cr^3 + 3Cr^2e + 3Cre^2 + Ce^3 \\ + Dr^4 + 4Dr^3e + 6Dr^2e^2 + 4Dre^3 + De^4 \\ + Er^5 + 5Er^4e + 10Er^3e^2 + 10Er^2e^3 + 5Ere^4 + \\ \quad Ee^5 \text{ \&c.} \end{array}\right\} = N$$

the ſum

$$P + ae + be^2 + ce^3 + de^4 + ge^5 = N$$

and $ae + be^2 + ce^3 + de^4 + ge^5 = N - P = f$.

Whence this

RULE.

Take r very near z, and let $r + e = z$, then ſubſtitute the powers of $r + e$ for thoſe of z, till you get $P + ae + be^2 + ce^3$ &c. $= N$, and $ae + bee$ &c. $= N - P = f$, which equation is to be reſolved by Prob. lxxxviii; or elſe the equation $ae + be^2 + ce^3$ $+ de^4$

$+ de^4$ &c. $= f$, is to be reſolved by ſome of the rules in the laſt problem, and the operation repeated if there be occaſion.

And here the coefficients a, b, c, d, &c. are moſt eaſily had from the terms, which compoſe the value of P; for we have $P = Ar + Br^2 + Cr^3 + Dr^4$ &c. Whence

$$a = \frac{Ar + 2Br^2 + 3Cr^3 + 4Dr^4 \text{ \&c.}}{r}$$

$$b = \frac{Br^2 + 3Cr^3 + 6Dr^4 + 10Er^5}{rr}$$

$$c = \frac{Cr^3 + 4Dr^4 + 10Er^5 \text{ \&c.}}{r^3},$$ and ſo on; where the numbers in a, are 1, 2, 3, 4, &c; in b, 1, 3, 2×3, 2×5. 3×5. 3×7, 4×7, 4×9, &c. in $c = 1$, 4, $\frac{5}{2}p$, $\frac{6}{3}q$, $\frac{7}{4}r$, $\frac{8}{5}s$, $\frac{9}{6}t$ &c. where p, q, r, s, t, &c. are the foregoing terms. And in finding a, b, c, &c. you muſt go through all the terms, till they grow very ſmall, and at laſt vaniſh. But you need not find above two or three of theſe coefficients a, b, c, &c. and each ſucceeding one may conſiſt of fewer places of figures.

Example.

Let $z - \frac{1}{2}z^2 + \frac{1}{2.3}z^3 - \frac{1}{2.3.4}z^4 + \frac{1}{2.3.4.5}z^5$ &c. $= \frac{2}{7}$.

Here by ſeveral trials z is found nearly $= \frac{1}{3}$; therefore put $r = \frac{1}{3}$, and $r + e = z$. Then $P = A + Br^2 + Cr^3$ &c. that is,

$r =$

$$\begin{array}{rcr} r & = & .333333 \\ +\frac{1}{6}r^3 & = & 6172 \\ +\frac{1}{120}r^5 & = & 34 \\ \hline & & .339539 \\ & & -.056071 \\ \hline & & P = .283468 \\ \hline \end{array} \qquad \begin{array}{rcr} -\frac{1}{2}r^2 & = & -.055555 \\ -\frac{1}{24}r^4 & = & -\quad 514 \\ -\frac{1}{720}r^6 & = & -\quad 2 \\ \hline & & -.056071 \end{array}$$

then $a = \dfrac{\begin{array}{rr} .333333 - & .111110 \\ +\ 18516 - & 2056 \\ +\ 170 & -\ 12 \end{array}}{\frac{1}{3}} = \dfrac{\begin{array}{r} .352019 \\ -.113178 \end{array}}{1} \times 3 =$ $.716523$. Also

$b = \dfrac{\begin{array}{rr} -.055555 + & .018516 \\ -\ 3084 + & 340 \\ -\ 30 & \end{array}}{\frac{1}{9}} = \dfrac{\begin{array}{r} -.058669 \\ +.018856 \end{array}}{1} \times 9$ $= -.3583$. Hence

$$.283468 + .71652e - .3583ee = \tfrac{2}{7} = .285714$$

and $.71652e - .3583ee = .002246$

and $1.9998e - ee = .00627$

$$e = \frac{.00627}{1.999 - e}$$

$$\begin{array}{rl} 1.999) & .006270\ (.00314 = e \\ -\ 3 & 5988 \\ \hline 1.996 & 282 \\ :: & 199 \\ \hline & 83 \\ & 79 \\ \hline & 4 \end{array}$$

$$\begin{array}{rl} \text{Then } r = & .33333 \\ e = & .00314 \\ \hline z = & .33647 \\ \hline \end{array}$$

or put $r = .33647$ for another operation.

SCHOLIUM.

If the ſeries breaks off, then it is no matter whether it converges or not. And in that caſe it coincides with the laſt problem, and may be ſolved by any of the rules therein.

And if e be very ſmall, the equation $ae + bee + ce^3$ &c $= f$, may be expeditiouſly ſolved by Prob. lxii. Rule 1, in which you need only uſe the three firſt terms; which will be ſhorter than taking new r. But that rule cannot ſo conveniently be applied to the given ſeries, becauſe it does not converge ſo faſt as this.

PROBLEM XCIV.

To extract the root in numbers of the infinite ſerie $Az + Bz^3 + Cz^5 + Dz^7$ &c. $= N$; *ſuppoſing i to converge faſt.*

Preparation.

Take r as near the root as it can be found by trials, and put $r + e = z$, and expunging z, w ſhall have,

$$\left.\begin{array}{l} Ar + Ae \\ +Br^3 + 3Br^2e + 3Bree + Be^3 \\ +Cr^5 + 5Cr^4e + 10Cr^3ee + 10Cr^2e^3 + 5Cre^4\text{\&c.} \\ +Dr^7 + 7Dr^6e + 21Dr^5ee + 35Dr^4e^3 + 35Dr^3e^4 \\ +Er^9 + 9Er^8e + 36Er^7ee + 84Er^6e^3 + 126Er^5e^4 \end{array}\right\} = N$$

the ſum

$$P + ae + bee + ce^3 + de^4 \text{ \&c.} = N.$$

an

and $ae + bee + ce^3 + de^4$ &c. $= N - P = f$,
Whence this

RULE.

Aſſume r by trials very near z, and $r + e = z$;
hen ſubſtitute the powers of $r + e$ for z, in the
iven ſeries, till you get $P + ae + be^2 + ce^3$ &c. $= N$,
And $ae + be^2 + cc^3$ &c. $= N - P = f$.

Vhere $P = Ar + Br^3 + Cr^5 + Dr^7 + Er^9$ &c.

$$a = \frac{Ar + 3Br^3 + 5Cr^5 + 7Dr^7 \text{ \&c.}}{r}$$

$$b = \frac{3Br^3 + 10Cr^5 + 21Dr^7 \text{ \&c.}}{rr}$$

$$c = \frac{Br^3 + 10Cr^5 + 35Dr^7 \text{ \&c.}}{r^3}$$

&c.

Vhere the numbers of a, are 1, 3, 5, 7 &c. of
; 3, 2×5, 3×7, 4×9, 5×11, 6×13, &c. And
ch ſeries is to be continued till the terms become
ery ſmall and vaniſh; which will happen in a lit-
e time, becauſe the given ſeries converges. The
rms of a, b, c, are eaſily had from the terms of
, as above, without much labour; then having
ot $ae + bee + ce^3$ &c. $= f$, in numbers; find the
ot e, by Prob. lxxxviii. or by ſome of the rules
Prob. xcii.

Example.

Let $y + \frac{1}{6}y^3 + \frac{3}{4.5}y^5 + \frac{3.5}{7.2.4.6}y^7 + \frac{3.5.7}{2.4.6.8}y^9$ &c. $= .698132$, *to find* y.

The ſeries abridged will be $y + \frac{\frac{1}{2}ryy}{3} + \frac{\frac{3}{4}Qyy}{5}$ +

$+\frac{\frac{5}{6}Ryy}{7}$ &c. $= .698132$; Q, R, S, being the numerators. By a few trials, y is nearly $= .6$, put $r = .6$; then

$$\begin{array}{lrr lrl}
r = & .600000 & & r = & .600000 & = Ar \\
Q = & .108000 & 3) & Q = & 36000 & = Br^3 \\
R = & 29160 & 5) & R = & 5832 & = Cr^5 \\
S = & 8748 & 7) & S = & 1249 & = Dr^7 \\
T = & 2756 & 9) & T = & 306 & \&c. \\
V = & 893 & 11) & V = & 81 & \\
W = & 295 & 13) & W = & 23 & \\
X = & 98 & 15) & X = & 6 & \\
Y = & 33 & 17) & Y = & 2 & \\
Z = & 11 & 19) & Z = & 1 & \\
\hline
 & & & & .643500 & = P.
\end{array}$$

Then $a = 1.2500$, the sum of the first column divided by r. $b = .585$; whence

$$.643500 + 1.250e + .585ee = .698132$$

and $1.250e + .585ee = .054632$

and $e = \frac{.0546}{1.25} = .043$ nearly.

and $e = \frac{.054632}{1.2500 + .585 \times .043} = \frac{.054632}{1.2500 + .0251}$

$= \frac{.054632}{1.275} = .04284$ more exactly.

add $r = .60000$

$.64284 = y$.

or take new $r = .6428$ for another operation.

PR

PROBLEM XCV.

o extract the roots of two given equations, containing two unknown quantities x, y; though never ſo compounded.

RULE.

By ſeveral trials find two near values of x and , *viz:* r and s, and put $r + e = x$, and $s + v = y$. And inſtead of the powers of x and y, put in hoſe of $r + e$, and $s + v$. Then involve all ſurds by the bynomial theorem (Prob. lviii.), alſo reduce logarithmic quantities to ſeries (Prob. xxxiv, lxxxv.), and the like for all compound quantities; ſo that at laſt the equations may conſiſt only of ſimple terms. And in doing this, reject all powers of e and v above the firſt, and alſo their products.

Then you will have two ſimple equations of e and v, which being reſolved, will give their values; and from hence x and y will be known. Then put new r and s for theſe values of x and y, and repeat the operation, which may be done as often as you pleaſe, till you get the roots as near as you have a mind. And the ſame form may ſtand and ſerve for all theſe operations.

Ex. 1.

Suppoſe $\sqrt{yy - xx} + \dfrac{2xy}{\sqrt{yy + 2x}} = 20 = b$

and $\dfrac{\log{:}x + \sqrt{xx + yy}{:}}{y} = 0.096 = c$.

Let $a = 2$. And by ſome trials we find x near 4, and y near 13; then put $r = 4$, $s = 13$, and by involution, and putting $r + e$ for x, and $s + v$ for y, we have

$\overline{ss - rr + 2sv - 2re}|^{\frac{1}{2}} + \overline{2rs + 2rv + 2se} \times$ $\overline{ss + 2sv + ar + ae}|^{-\frac{1}{2}} = b$ and $\frac{\text{log:}\, r + e + \overline{rr + 2re + ss + 2sv}|^{\frac{1}{2}}}{s + v} = c.$

but $\overline{ss - rr + 2sv - 2re}|^{\frac{1}{2}} = \sqrt{ss - rr} + \frac{sv - re}{\sqrt{ss - rr}}$

and $\overline{ss + ar + 2sv + ae}|^{-\frac{1}{2}} = \frac{1}{\sqrt{ss + ar}} - \frac{2sv + ae}{2 \times \overline{ss + ar}|^{\frac{3}{2}}}$;

alſo $\overline{rr + ss + 2re + 2sv}|^{\frac{1}{2}} = \sqrt{rr + ss} + \frac{re + sv}{\sqrt{rr + ss}}$.

Put $dd = ss - rr$, $ff = ss + ar$, $gg = ss + rr$. Then we have $d + \frac{sv - re}{d} + \overline{2rs + 2rv + 2se} \times \overline{\frac{1}{f} - \frac{2sv + ae}{2f^3}}$ $= b$, that is, (1) $\overline{\frac{2s}{f} - \frac{r}{d} - \frac{ars}{f^3}} \times e + \overline{\frac{s}{d} + \frac{2r}{f} - \frac{2rss}{f^3}}$ $\times v = b - d - \frac{2rs}{f}$.

Again, $\frac{\text{log:}\, r + e + \overline{rr + ss + 2re + 2sv}|^{\frac{1}{2}}}{s + v} = c =$ $\frac{\text{log:}\, r + e + g + \frac{re + sv}{g}}{s + v}$, and log: $r + e + g + \frac{re + sv}{g}$ $= \overline{s + v} \times c$. Put $t = r + g$, $l =$ log: t. $m. = .4342945$, then (Prob. lxxxiv.) $l + \frac{mte + msv}{tg} = cs + cv$, which reduced, is (2) $mte + msv = tg \times \overline{cs - l}$. Then numbers being

$- cgt$

ſubſtituted in theſe two equations, give

(1) $1.588e + 1.075v = - 0.190$
(2) $7.643e - 17.3326v = + 0.59536$;

An

And theſe equations being reſolved, give $e=-.0743$, and $v=-.0671$; whence $r+e$ or $x=3.926$; and $s+v$ or $y=12.933$.

Or for another operation, put $r=3.926$ and $s=12.933$, and finding new values for d, f, g, t, and l, you will have two equations, which will give e and v more exactly.

Ex. 2.

Let $x+\log{:}y=b=8.7679114.$
and $y+\log{:}x=c=3.4760046.$

By a few trials, we find x nearly $=8$, and $y=2\frac{1}{2}$. Put $r=8$, and $r+e=x$; alſo $s=2\frac{1}{2}$, and $s+v=y$. Alſo $M=.4342945$; then we have

$$r+e+l{:}\overline{s+v}=b=r+e+l{:}s+\frac{Mv}{s}\ \text{(Pr. 84.)}$$

$$\text{and } s+v+l{:}\overline{r+e}=c=s+v+l{:}r+\frac{Me}{r}.$$

Theſe equations reduced become

$$e+\frac{Mv}{s}=b-r-l{:}s.$$

$$v+\frac{Me}{r}=c-s-l{:}r.$$

And put into numbers are

$$e+.1737v=.3700.$$
$$\text{and } v+.0543e=.0730.$$

Which equations being reſolved give $e=.3608$, and $v=.0535$; whence

$$\begin{array}{rl} r = & 8.0000 \\ +e = & .3608 \\ \hline x = & 8.3608 \end{array} \qquad \begin{array}{rl} s = & 2.5 \\ +v = & .0535 \\ \hline y = & 2.5535 \end{array}$$

Again, put $r = 8.3608$, and $s = 2.5535$, for another operation; whence will be found $\frac{M}{s} = .170078$, $\frac{M}{r} = .051944$; and $b - r - l : s = -.0000245$, $c - s - l : r = .0002568$, and from thence will ariſe theſe two equations,

$$\begin{aligned} e + .170078v &= -.0000245, \\ \text{and } v + .051944e &= .0002568. \end{aligned}$$

Which being reſolved, give $e = -.0000688$, and $v = .0002604$; therefore

$$\begin{array}{rl} r = & 8.3608000 \\ +e = & -.0000688 \\ \hline x = & 8.3607312 \end{array} \qquad \begin{array}{rl} s = & 2.5535000 \\ +v = & .0002604 \\ \hline y = & 2.5537604. \end{array}$$

SECT. IX.

The Geometrical Conſtruction of Equations.

THE *conſtruction* of equations, is the drawing rights lines or curves, after ſuch a manner, as by their interſections, to give the roots of the equation propoſed. This method is uſed for avoiding the tedioufneſs of computation; and is exact enough for finding two or three of the firſt figures of the root, but not more. For where great exactneſs is required, we are not to truſt to a conſtruction by lines; but make a computation in numbers, to find the root.

In geometrical conſtructions, the ſimpleſt is always to be made uſe of, or that by which we can come the ſhorteſt way, to the roots of the equation propoſed.

But ſince the extraction of roots by converging ſeries, is now brought to ſo great perfection; geometrical conſtructions are almoſt laid aſide. Therefore I intend to trouble the reader only with the ſhorteſt methods of conſtructing equations as far as the fourth power. When we come to higher powers, there is ſo much trouble and difficulty in drawing the lines proper for them, that their interſections cannot be depended on; and one may ſooner extract the root in numbers.

PROBLEM XCVI.

To conſtruct a ſimple equation.

RULE.

1. When there are ſeveral ſimple quantities, connected by the ſigns + and —. From a certain

point, draw a right line, from which point ſet all affirmative quantities one way, one adjoining to another; from the laſt point, ſet all the negative quantities the contrary way, adjoining to each other as before. Where the laſt ends, the diſtance from thence, to the firſt point, gives the ſum (or difference) of all; which is affirmative or negative, according as it lies on the affirmative or negative ſide of the firſt point aſſumed.

2. When you have the ſquare root of two quantities, find a mean proportional between them, by Prob. 16. B. VIII. Geometry.

3. To reduce two compound quantities to the ſame deſignation. By Prob. 15. B. VIII. Geometry, find one or more proportionals thus; ſay, as the firſt letter of the firſt quantity, to the firſt in the ſecond, ſo the ſecond in the ſecond to that fourth proportional. Again, as the ſecond letter in the firſt quantity to the third letter in the ſecond; ſo the fourth proportional laſt found, to another fourth proportional. Proceed thus till all the letters in one quantity be exhauſted

Note, when any term is of too low a dimenſion, make 1 to be one of the factors, as oft as it is wanted. And when you have ſeveral ſimple quantities, add them into one, by Art. 1.

4. For many compound quantities, reduce them all to the ſame deſignation by Art. 3.

Ex. 1.

Suppoſe $a + b - c = x$.

Fig. 3. Draw the line DAB, and from the fixt point A, ſet off $AB = a$, and make $BC = b$, both forward; laſtly, make $CE = c$, backward. Then $+ AE = x$.

Ex.

Fig.

Ex. 2.

Let $ax = bc$, *to find* x.

Make AB (a) : AC (b) : : AD (c) : AE 4.
$= x$, (by Prob. 15. Geom.)

Ex. 3.

Suppoſe $2abcx = 5defg$.

$$\begin{array}{lcccccc} \text{Make as} & 2a & : & 5d & :: & e : m & \text{(Pr. 15. Geom.)} \\ \text{and} & b & : & f & :: & m : n & \\ \text{and} & c & : & g & :: & n : p & \\ \hline \text{then} & 2abc & : & 5dfg & :: & e : p & \end{array}$$

$$\text{and } 2abcp = 5defg = 2abcx$$

$$\text{or } p = x.$$

Ex. 4.

Let $abx - f\sqrt{bc} \times x = dd\sqrt{ac - bd}$.

By Prob. 15. Geom. make $a : b :: d : m$; then $bd = am$; and $\sqrt{ac - bd} = \sqrt{ac - am}$, make $c - m = n$, then $\sqrt{ac - bd} = \sqrt{an}$. Find p a mean proportional between a and n, and q a mean between b and c, (Prob. 16. Geom.) then the given equation becomes $abx - fqx = ddp$.

Reduce theſe three terms to the ſame deſignation, thus $a : f :: q : r$, whence $fq = ar$, in like manner $dd = as$; then the equation is $abx - arx = asp$, or $bx - rx = sp$. Put $b - r = t$, then $tx = sp$, and $t : s :: p : x$ required.

Ex. 5.

Let $2abcdd - eefgh + 3kllmn = 4qrstz - 5noplz$, *to find* z.

Reduce all the quantities to the ſame deſignation, then

Fig. 4.

$$4qrsx = 5nopl$$
$$4qrstv = 2abcdd$$
$$4qrstw = eefgh$$
$$4qrsty = 3kllmn.$$

then the equation becomes

$$4qrstv - 4qrstw + 4qrsty = 4qrstz - 4qrsxz.$$

that is, $tv - tw + ty = tz - xz$

Put $v - w + y = A$, $t - x = B$, then $At = Bz$, or $B : A :: t : z$.

P R O B. XCVII.

To conſtruct a quadratic equation.

1 RULE.

If it is a pure quadratic; reduce the quantities concerned therein to the ſame deſignation (Prob. xcvi. Art. 3.) by which means ſurds will be denoted by ſimple quantities, and at laſt you will get all the known quantities equal to a known ſquare, whoſe ſide is the root.

Ex. 1.

Suppoſe $yy = ab - \frac{cdd}{b} + d\sqrt{aa - bc}$.

Make $b : c :: d : m$, (Prob 15. Geom.) then $cd = bm$, and $\frac{cdd}{b} = \frac{bmd}{b} = md$. Alſo make $a : b :: c : n$, then $bc = an$; whence $yy = ab - md + d\sqrt{aa - an}$.

Let p be a mean between a and $a - n$, (Prob. 16. Geom.) then $\sqrt{aa - an} = p$, whence $yy = ab - md + dp$.

Let $d : a :: b : q$, then $ab = dq$.

then $yy = dq - dm + dp$.

Laſtly, put $q - m + p = r$, and find s a mean between d and r; then $yy = dr = ss$, and $y = s$.

2 RULE.

Fig.

2 RULE.

5. In adfected quadratics, reduced to this form $aa \pm ba = nn$. Draw a right line AD, then take any point C; and make CB$=\frac{1}{2}b$, towards the right hand if $+b$, or towards the left, if $-b$. Erect the perpendicular BF $= n$. From the center C through F, deſcribe the circle AFD, to cut AD. Then (BD, BA) the diſtances of B, from the interſections A, D, are the two roots, the affirmative to the right hand, the negative to the left of B.

Ex. 2.

Let $aa + 3a = 10$.

5. Draw the line AD, make CB $= 1\frac{1}{2}$ on the right; find a mean proportional between 1 and 10, ſet it in the line BF, perpendicular to AB, with the radius CF deſcribe the circle AFD; then $a =$BD$= +2$, and $a =$BA$= -5$, the two roots required.

Ex. 3.

Suppoſe $aa - 3a = 10$.

6. Draw the line AD, make CB (on the left of C) $= 1\frac{1}{2}$, find a mean proportional between 1 and 10; at B erect the perpendicular BF, and make BF $=$ the mean; with the radius CF deſcribe the circle AFD; to cut AD in A and D; then $a =$BD$= +5$, and $a =$BA$= -2$, the roots required.

3 RULE.

7. In ſuch quadratic equations as may be reduced to this form, $aa \pm ba = -nn$. From any point C as a center, in the right line BD, with radius $\frac{1}{2}b$, deſcribe the circle BFD, erect a perpendicular at D on the right, if it be $+b$, or on

Fig. on the left at B, if it is $-b$; whoſe length is
7. BA $= n$. Through A draw AFG parallel to BD, to interſect the circle in F and G; then AF and AG are the two roots of the equation; which are affirmative, if they lie towards the right hand from A; or negative, if on the left.

Note, if the parallel does not cut the circle, or touch it, the equation is impoſſible.

Ex. 4.

Suppoſe $aa + 7a = -10$;

With the radius $3\frac{1}{2}$, and center C, deſcribe
8. the circle BFD. At the end of the diameter D, on the right, raiſe the perpendicular DA, a mean between 1 and 10. Through A draw AFG parallel to the diameter BD, to cut the circle in F and G; and AF, AG, being on the left from A, are two negative roots; $a = \text{AF} = -2$, and $a = \text{AG} = -5$.

Ex. 5.

Let $aa - 7a = -10$.

With the radius $\text{CB} = 3\frac{1}{2}$, and center C,
7. deſcribe the circle BFD; at the end B, of the diameter BD on the left, raiſe the perpendicular BA, equal to the mean between 1 and 10. Through A, draw AFG parallel to the diameter BD, to cut the circle in F and G; then AF, AG, lying on the right hand from A, are the two affirmative roots; and $a = \text{AF} = 2$, and $a = \text{AG} = 5$.

4 RULE.

When the unknown quantity is higher than the ſquare, and the index in one term double to that in the other; it may be brought to ſome of the foregoing forms, whoſe higheſt term is a ſquare.

a ſquare. Aſſume an unknown quantity, whoſe Fig.
rectangle with ſome given quantity, is equal to the ſquare of the unknown quantity propoſed; for this ſubſtitute that rectangle; and you will have an equation as required.

Ex. 6.

Let $z^4 - bzz = n$.

Aſſume $dx = zz$, then by ſubſtitution, $ddxx - bdx = n$, and $xx - \frac{b}{d}x = \frac{n}{dd}$. Let $d : b :: 1 : p$; then $b = dp$; alſo make $d : n :: 1 : q$, and $d : 1 :: q : r$, then $dd \,.\, n :: 1 : r$, and $ddr = n$. And the leaſt equation becomes $xx - \frac{dp}{d}x = \frac{ddr}{dd}$, that is, $xx - px = r$, which is to be conſtructed by ſome of the former rules.

To demonſtrate theſe rules. Let $aa + ba$
$= nn$. Here we have CB $= \frac{1}{2}b$, BF $= n$, and 5.
if BD $= a$, then CD or CF $= a + \frac{1}{2}b$, and
CF^2 $=$ CB^2 $+$ BF^2, that is, $\overline{a + \frac{1}{2}b}|^2 = \frac{1}{4}bb + nn$.
But if BA $= -a$, then CA or CF $= -a - \frac{1}{2}b$,
and $\overline{-a - \frac{1}{2}b}|^2 = \frac{1}{4}bb + nn$. In both caſes $aa +$
$ba = nn$.

Again, if $aa - ba = nn$, we have as before
CB $= \frac{1}{2}b$, BF $= n$, and if BD $= a$, then CD or 6.
CF $= a - \frac{1}{2}b$, and $\overline{a - \frac{1}{2}b}|^2 = \frac{1}{4}bb + n$.
But if AB $= -a$, then AC or FC $= -a$
$+ \frac{1}{2}b$, and $\overline{-a + \frac{1}{2}b}|^2 = nn + \frac{1}{4}bb$, in both caſes
$aa - ba = nn$.

Again, if $aa - ba = -nn$; here BC $= \frac{1}{2}b$,
BA $= n$, and AG $=$ BD $-$ AF, therefore if 7.
AF $= a$, then AG $= b - a$, and AG $\times$
AF $=$ AB^2, that is, $\overline{b - a} \times a = nn$.

Fig. But if $AG = a$, then $AF = BD - AG = b - a$, and $AF \times AG = AB^2$, or $\overline{b-a} \times a = nn$. In both caſes $aa - ba = -nn$.

8. Laſtly, when $aa + ba = -nn$, then $BC = \frac{1}{2}b$, $BA = n$, as before; then if $AF = -a$, then $AG = BD - AF = b + a$, and $AF \times AG = AD^2$, or $-a \times \overline{b +} \ = nn$.

But if $AG = -a$, then $AF = BD - AG = b + a$, and $AF \times AG = AD^2$, or $\overline{b + a} \times -a = nn$. In both caſes $aa + ba = -nn$.

PROB. XCVIII.

To conſtruct cubic and biquadratic equations.

9. To conſtruct a cubic equation, that has all its roots real, by a circle. Let the radius $OB = R$, ſine $EF = s$, GH the ſine of the arch GB or 3BE. Then by trigonometry, $3s - \frac{4s^3}{RR} = GH$. Draw CD parallel to AB, and put $SF = c$, $ES = x$, $GH = b$, then $c + x = s$, whence $3 \times \overline{c + x} - \frac{4}{RR} \times \overline{c+x}|^3 = b$, this reduced gives

$$\begin{aligned} x^3 + 3cx^2 + 3ccx &+ c^3 = 0. \\ -\tfrac{3}{4}RR &+ \tfrac{1}{4}bRR \\ &- \tfrac{3}{4}cRR \end{aligned}$$

Suppoſe this cubic equation be given, $x^3 + px^2 + qx + r = 0$. Comparing this with the former, and equating the coefficients, we have $p = 3c$, and $c = \frac{1}{3}p$. Alſo $q = 3cc - \frac{3}{4}RR = \frac{1}{3}pp - \frac{3}{4}RR$, whence $R = \frac{2}{3}\sqrt{pp - 3q}$, and $r = c^3 + \frac{1}{4}bRR - \frac{3}{4}cRR$; whence $b = \frac{gr - pq}{pp - 3q} + \frac{2}{3}p$. Hence ariſes the following

1 RULE.

Fig.

1 RULE.

Having the equation $x^3+px^2+qx+r=0$. given; 10.

1. With the radius $\frac{2}{3}\sqrt{pp-3q}$, deſcribe the circle BGAK.
2. Draw the diameter AB, and CD parallel to it, at the diſtance of $\frac{1}{3}p$; above it, if it be $+p$, but below it, if $-p$.
3. Draw alſo ZG parallel to AB, at the diſtance $\frac{9r-pq}{pp-3q}+\frac{2}{3}p$, above it, if it is affirmative; or below it, if negative. Let it cut the circle in G.
4. Take the arch $\text{BP}=\frac{1}{3}\text{BG}$; and make $\text{PQ}=\text{QK}=\text{KP}$.
5. From the points P, Q, K, let fall perpendiculars, upon the line CD, which will be the roots of the equation; the affirmative above the line, and the negative below it.

SCHOLIUM.

If $3q$ be greater than pp, the equation is impoſſible; for in this caſe the equation has two impoſſible roots.

Alſo if $p=0$, then the radius of the circle $\text{OB}=\frac{2}{3}\sqrt{-3q}$; and CD coincides with AB; and the diſtance of ZG from AB is $-\frac{3r}{q}$. And if q is affirmative, the equation is impoſſible. Theſe conſtructions are extreamly eaſy.

Ex. 1.

Let $x^3+9x^2-22x-120=0$.

Here the radius $\text{OB}=\frac{2}{3}\sqrt{pp-3q}=$ 11. $\frac{2}{3}\sqrt{81+66}=8.0829$, and $\frac{1}{3}p=3$, the diſtance of CD, above AB.

X And

Fig. And $\frac{9r-pq}{pp-3q}+\frac{2}{3}p=\frac{-1080+198}{81+66}+\frac{2}{3}\times 6$

11. $=-6+6=0$, the diſtance of GZ from AB; therefore ZG coincides with AB; and the arch BG and alſo its third part is 0, and P falls on B; and making PQ=QK=KP, and letting fall perpendiculars on CD, we ſhall have PS $=-3$, QT $=+4$, and KT$=-10$, the three roots required.

Ex. 2.

Suppoſe $x^3-17x^2+82x-120=0$.

12. The radius OB $=\frac{2}{3}\sqrt{289-246}=4.37$. $\frac{1}{3}p=-5.66$, the diſtance of CD below AB, and $\frac{9r-pq}{pp-3q}+\frac{2}{3}p=\frac{-1080+1394}{289-246}-\frac{34}{3}=$ $7.302-11.333=-4.031$, the diſtance of GZ below AB. Take BP the third part of BG, and making PQ=QK=KP; and meaſuring the perpendiculars upon CD, we have PS $=+4$, QT $=+10$, and KV $=+3$, the roots of the equation.

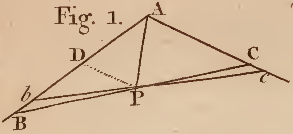
Ex. 3.

Let $y^3-13y+12=0$.

13. In this example $p=0$, therefore CD coincides with AB; and radius OB$=\frac{2}{3}\sqrt{-3q}=$ $\frac{2}{3}\sqrt{39}=4.18$; and $-\frac{3r}{q}=\frac{-36}{-13}=2.77$ the diſtance of ZG above AB. Take arch BP$=\frac{1}{3}$ arch BG, and make PQ=QK=KP; and let fall perpendiculars on AB, then PS $=+1$, QT $=+3$, and KV $=-4$, the three roots required.

14. Cubic equations may alſo be conſtrued by a cubic parabola and a right line. Let FVAC be a cubic parabola, whoſe latus rectum is 1. Draw

Fig. 1.


A
D
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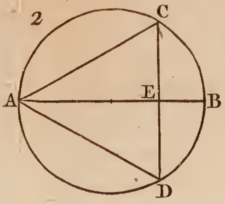
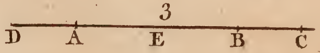

2
C
A
E
B
D


3
D
A
E
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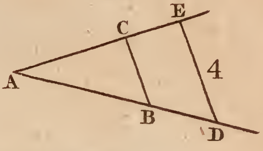
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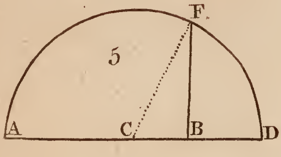
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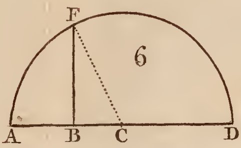
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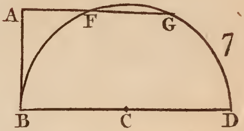
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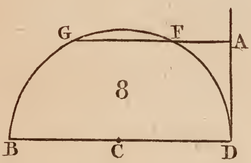
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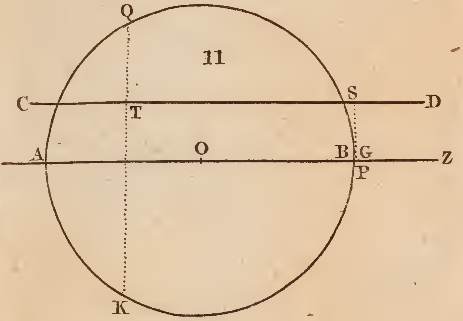
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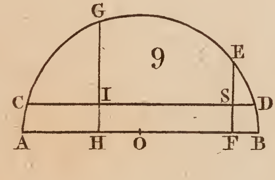
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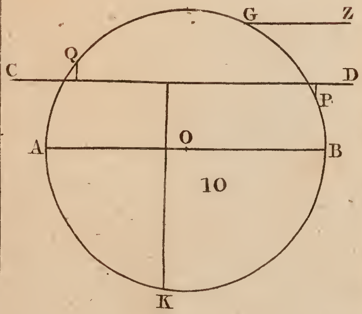
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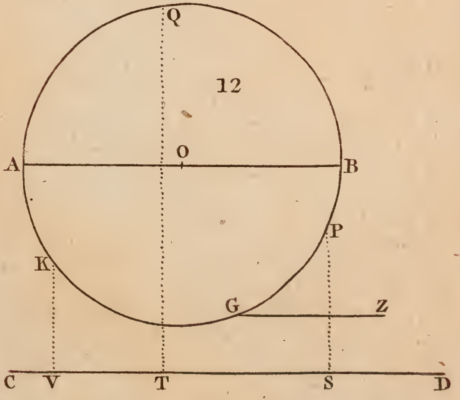
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Fig. 14. Draw VE the tangent at the vertex, perpendicular to the axis VS, and BI parallel to it, and SBC perpendicular to AB.

Then put $VH = b$, $VD = c$, VI or $SB = n$, $BC = a$, $VS = x$, then $SC = n + a$, and by the property of the parabola $VS = SC^3$, or $x = \overline{n + a}^3$. By ſimilar triangles, VH (b) : VD (c) : : SH $(x - b)$: $SC = \frac{cx - cb}{b}$; and BC or $a = \frac{cx - bc}{b} - n$, and $ba = cx - cb - nb$, whence $cx - ba = cb + nb$, or $x - \frac{b}{c}a = b + \frac{nb}{c}$, that is (expunging x) $n^3 + 3n^2a + 3na^2 + a^3 - \frac{b}{c}a = b + \frac{nb}{c}$, which reduced is

$$\begin{array}{llll} a^3 + + 3na^2 & + 3nna & + n^3 & = 0. \\ & - \frac{b}{c}a & - b & \\ & & - \frac{nb}{c} & \end{array}$$

Let $a^3 + pa^2 + qa + r = 0$, be any cubic equation. By comparing them, and equating the like terms; we have $3n = p$, and $n = \frac{1}{3}p$. Alſo

$3nn - \frac{b}{c} = q$, and $\frac{b}{c} = 3nn - q = \frac{1}{3}pp - q$.

Again, $n^3 - b - \frac{bn}{c} = r$, or $\frac{1}{27}p^3 - b - \frac{1}{3}p \times \overline{\tfrac{1}{3}pp - q} = r$, whence $b = \frac{1}{3}pq - \frac{2}{27}p^3 - r$.

And ſince $\frac{b}{c} = \frac{1}{3}pp - q$, $c = \frac{b}{\frac{1}{3}pp - q} = \frac{pq - \frac{2}{9}p^3 - 3r}{pp - 3q}$. Whence we have the following conſtruction, by this

Fig. 14.

2 RULE.

Given the equation $a^3+pa^2+qa+r=0$.

1. With the parameter 1, and the axis VS, deſcribe the cubic parabola FVAC, draw the diameter RAB, diſtant $\frac{1}{3}p$ from the axis VS, to the right hand, if affirmative; and draw the tangent at the vertex IVD.
2. In the axis VS take $VH=\frac{1}{3}pq-\frac{2}{27}p^3-r$, downwards, if affirmative.
3. In the tangent IVD, take $VD=\frac{VH}{\frac{1}{3}pp-q}$ $=\frac{pq-\frac{2}{9}p^3-3r}{pp-3q}$, to the left, if affirmative.
4. Through the points D, H, draw the righ line FDHC, to cut the parabola. From all the points of interſection, let fall perpendiculars on the diameter AB, which will be the roots of the equation; thoſe on the right hand of AB affirmative; theſe on the left, negative.
5. When any of the aforeſaid quantities are negative, they muſt be laid the contrary way to what is directed above (Art. 1, 2, 3).

SCHOLIUM.

If the ſecond term be wanting, $p=0$, and AB coincides with VS; and then $VH=-r$, and $VD=\frac{r}{q}$.

If the numbers given in the equation, be too great for your parabola; the equation is eaſily changed into another with leſs numbers, by Prob. xlii.

Ex

Ex. 4.

Let the equation be $a^3 + \underset{p}{1.8}a^2 - \underset{q}{.5125}a - \underset{r}{1.05} = 0$.

The parabola being deſcribed, we have $VI = \frac{1}{3}p = .6$, the diſtance of IB from the axis VS, on the right; and $VH = \frac{1}{3}pq - \frac{2}{27}p^3 - r = .3105$, taken downwards from the vertex V. 14.
And $VD = \dfrac{VH}{\frac{1}{3}pp - q} = .195$, on the left from V.
Through D, H draw the right line FDHC, cutting the parabola in F, G, C; from which points letting fall perpendiculars on AB, we have $FR = -1.75$, $GL = -.8$, and $CB = +.75$, the three roots of the equation.

Ex. 5.

Let $x^3 - \frac{7}{4}x + \frac{3}{4} = 0$.

Here $p = 0$, and therefore AB coincides with VS; then make $VH = -r = -\frac{3}{4}$, upward; and $VD = \dfrac{r}{q} = -\frac{3}{7}$, to the right hand. 15.
Then through H, D draw the line FDC, to cut the parabola, in F, G, C; from which letting fall perpendiculars on VS, we have $FR = -1\frac{1}{2}$, $GL = +\frac{1}{2}$, and $CB = +1$, the three roots required.

Cubic and biquadratic equations may alſo be conſtructed by the common parabola. Let FVAC be a parabola, VS its axis, AB a diameter parallel to it. EA, and SBC two ordinates perpendicular to VS. Draw alſo HD perpendicular, and HQ parallel to VS; draw HC. 16.

Fig. 16. Put EA or SB $= c$, AD $= d$, DH $= g$, HC $= f$, and BC $= a$. Then QC $= g + a$, latus rectum of the figure $= 1$.

Then by the property of the figure AB $=$ FB $\times$ BC $= 2ca + aa$, and DB or HQ $= 2ca + aa - d$, and $HC^2 = HQ^2 + QC^2$, that is, $ff = a^4 + 4ca^3 + 4ccaa + dd - 4cad - 2daa + gg + 2ga + aa$; which reduced is,

$$\begin{array}{llllll} a^4 + 4ca^3 & + 4cca^2 & + 2ga & + gg & = 0. \\ & - \ 2d & - 4dc & + dd & \\ & + \ 1 & & - ff & \end{array}$$

Let $a^4 + pa^3 + qa^2 + ra + s = 0$, be any biquadratic equation; compare this term by term with the other; to find the values of the quantities c, d, f, g. Then we have $4c = p$, and $c = \frac{1}{4}p$.

Again, $4cc - 2d + 1 = q$, and $2d = 4cc + 1 - q = \frac{1}{4}pp + 1 - q$, and $d = \frac{\frac{1}{4}pp + 1 - q}{2}$.

Again, $2g - 4cd = r$, and $g = \frac{4cd + r}{2} = \frac{pd + r}{2}$.

Laſtly, $gg + dd - ff = s$, and $ff = gg + dd - s$. From hence ariſes the following conſtruction.

3 RULE.

Having the equation,

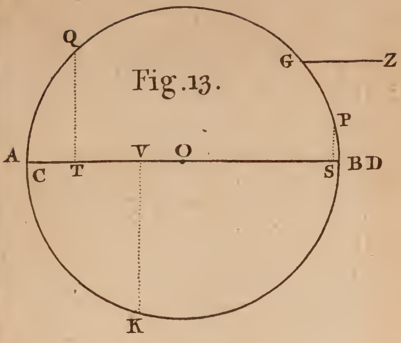
$$a^4 + pa^3 + qa^2 + ra + s = 0.$$

$$\text{or} \quad a^3 + pa^2 + qa + r = 0.$$

1. Deſcribe a parabola FVAC, whoſe parameter is 1, and axis VS. Draw the diameter AB at the diſtance of $\frac{1}{4}p$ from the axis, on the right hand; if p is affirmative. Then for the central rule.

2. From

Fig. 13.



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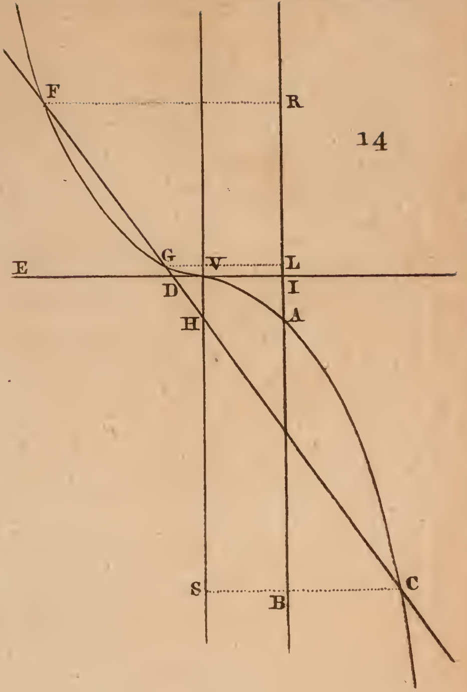

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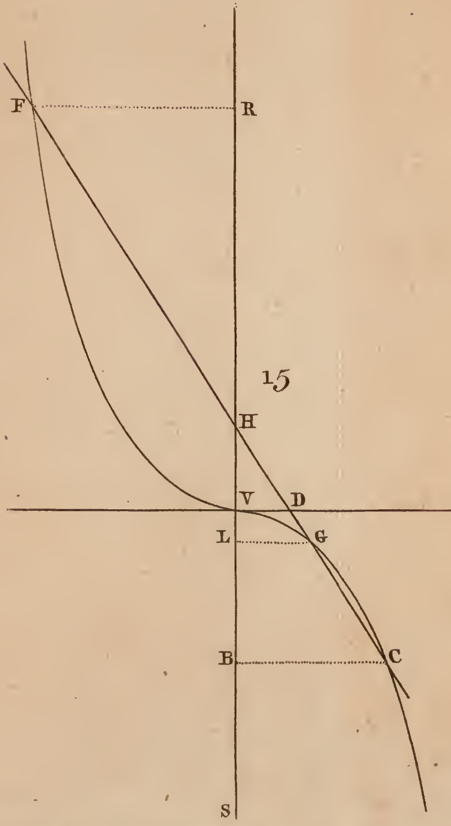

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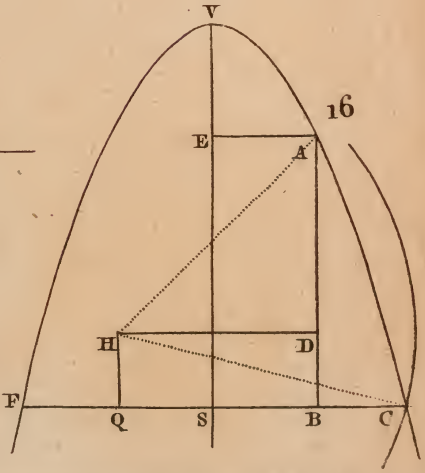


Pl. II. *pa. 325*

2. From A, the top of the diameter, take $AD = \frac{\frac{1}{4}pp + 1 - q}{2}$, downwards, if affirmative. Fig. 16.

3. From D in the perpendicular DH, take $DH = \frac{p \times AD + r}{2}$ towards the left, if affirmative.

4. But when any of theſe quantities are negative, ſet them the contrary way.

5. From the center H with the radius $\sqrt{AD^2 + DH^2 - s} = \sqrt{HA^2 - s}$, deſcribe a circle which will cut the parabola in ſeveral points as C.

6. From the points of interſection, let fall perpendiculars upon the diameter AB, and theſe will be the roots of the equation; theſe (BC) on the right ſide of AB, are affirmative roots; and on the left ſide, negative. And there are always as many real roots, as there are points of interſection; and the reſt are impoſſible.

SCHOLIUM.

If the ſecond term be wanting; then $p = 0$, and the diameter AB coincides with VS. Then alſo $AD = \frac{1 - q}{2}$, and $DH = \frac{1}{2}r$.

In cubics s is wanting, and then the radius HC becomes $= HA$.

If the numbers or coefficients be too large for your parabola, you muſt transform the equation, into another to ſuit your parabola, by Prob. xlii. and then conſtruct it; and laſtly, reſtore the true roots.

Fig.

Ex. 6.

Suppoſe $y^3 + 20y^2 - 500y - 6000 = 0$.

The numbers being too large, put $x = \frac{1}{10}y$, or $y = 10x$; then the equation becomes $1000x^3 + 2000x^2 - 5000x - 6000 = 0$, that is, $x^3 + 2x^2 - 5x - 6 = 0$, where the numbers are ſmall.

37. The parabola FVC being deſcribed, make $EA = \frac{1}{4}p = \frac{1}{2}$, on the right, and draw AB parallel to the axis VS.

Make $AD = \frac{\frac{1}{4}pp + 1 - q}{2} = \frac{2 + 5}{2} = 3\frac{1}{2}$, downwards. Draw DH perpendicular to AD, and make $DH = \frac{p \times AD + r}{2} = \frac{2 \times 3\frac{1}{2} - 6}{2} = \frac{1}{2}$, to the left. From the center H, with radius HA, deſcribe a circle, cutting the parabola in R, A, C, F; from which letting fall perpendiculars on AB; we have $RA = -1$, $BC = +2$, $FB = -3$, the roots of the equation $x^3 + 2x^2 - 5x - 6 = 0$, and multiplying by 10, we have -10, $+20$, and -30, for the roots of the given equation $y^3 + 20y^2 - 500y - 6000 = 0$.

Ex. 7.

Let $x^4 - 1.75x^3 - 4.625x^2 + 4.875x + 6.75 = 0$.

38. Deſcribe the parabola FVC, and draw the axis VS, and make $EA = \frac{1}{4}p = -.44$, to the left, and draw AB parallel to the axis, make $AD = \frac{\frac{1}{4}pp + 1 - q}{2} = 3.19$ downwards. Draw DH perpendicular to AB, and make $DH = \frac{p \times AD + r}{2} = -.36$, to the right.

From the center H, with the radius Fig. $\sqrt{AD^2 + DH^2 - 6.75} = 2$, deſcribe a circle 18. cutting the parabola in F, R, G, C; from which drawing perpendiculars to AB, we have $RO = -1$, $FB = -1\frac{1}{2}$, $GI = 2$, $CP = 2.25$, the roots required.

Ex. 8.

Given the equation $x^4 - 1.5x^3 + 5x^2 - 9x - 6 = 0$.

Deſcribe the parabola RVC to the axis VS, 19. and make $EA = \frac{1}{4}p = -.375$, to the left, and draw AB parallel to VS. Take $AD = \frac{\frac{1}{4}pp + 1 - q}{2} = -1.72$ upwards. Then (perpendicular to AD) take $DH = \frac{p \times AD + r}{2} = -3{,}21$, to the right. With the center H, and radius 4.39 ($= \sqrt{AD^2 - DH^2 + 6}$), deſcribe a circle, to interſect the parabola; from the points of interſection, letting fall perpendiculars on AB, gives the roots, $RO = -.5$, and $CB = +2$. The other two roots are impoſſible, which is known from this, that the circle interſects the parabola in no more points than theſe two.

Ex. 9.

Let $x^4 - 5\ 67x^2 + .806x + 3.864 = 0$.

Here $p = 0$, therefore deſcribe the parabola 20. FAC, whoſe axis is AS; and make $AD = \frac{1 - q}{2} = 3.335$, downwards; and $DH = \frac{1}{2}r = .403$, perpendicular to AD, to the left. With radius $\sqrt{AD^2 + DH^2 - 3.864} = 2.72$, deſcribe a circle cutting the parabola, in R, G, C, F; and

Fig. and the perpendiculars from theſe points up-
20. on AS, give RE$=-.8$, GI$=+1$, CB$=+2.1$, and F$s=-2.3$, the roots of the equation.

SCHOLIUM.

Geometrical equations may be conſtructed by lines as well as by numbers. For proper lines may be found for the coefficients, by proceeding according to Prob. xcvi; and ſo the whole may be done geometrically.

Quadratic equations, whoſe general form is $a^2+pa+q=0$, may alſo be conſtructed by the laſt rule; and then r and s will be $=0$; but the method of conſtructing by the circle, is eaſier.

4 RULE.

Any cubic or biquadratic equation $x^4+px^3+qx^2+rx+s=0$, may be conſtructed mechanically thus:

21. 1. Upon a plain ſmooth wall, draw a horizontal line AB, and CD perpendicular to it, and take CP $=\frac{1}{4}p$, to the left hand if p is affirmative. Hang a thread and plummet EPF to any point E, in the perpendicular EP; make a knot in the thread at n, and tie the other ends ſo to the fixt point E, that Pn may be $=\frac{1}{2}$. Then with a pin or the point of a compaſs, move the thread EF ſideways toward CD, till the knot n falls in the point C; mark the point D in the line CD, where the pin is, when that happens.

2. From D take DG$=\frac{\frac{1}{4}pp+1-q}{2}$ $(=d)$, downwards, if affirmative. And in the perpendicular GH, take GH$=\frac{dp+r}{2}$, to the left,

if

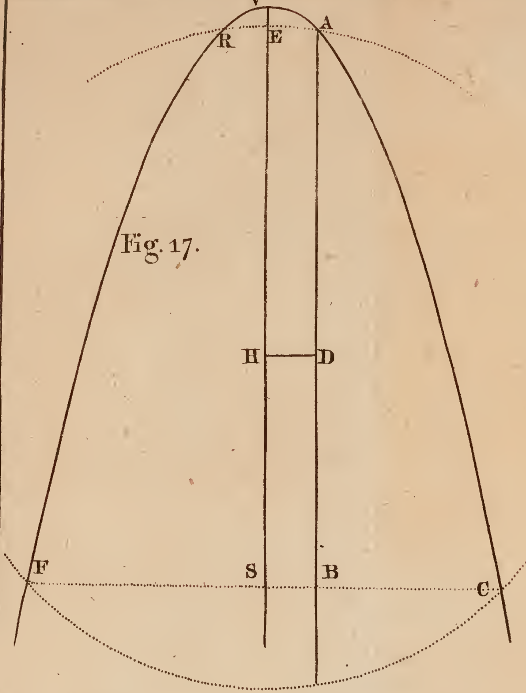
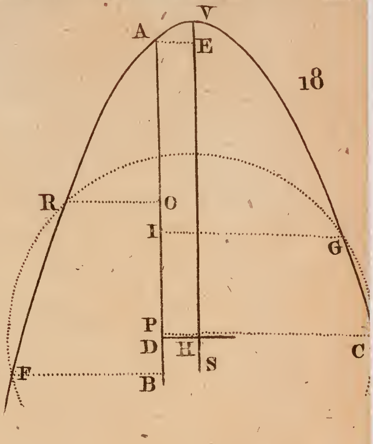


Fig. 17.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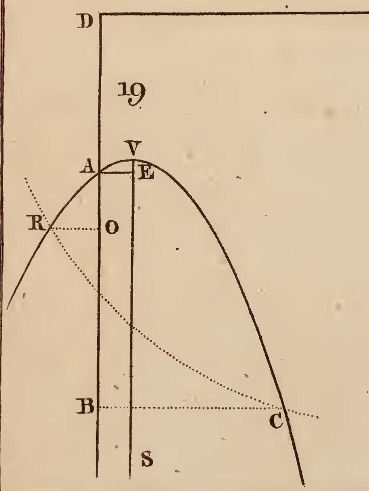
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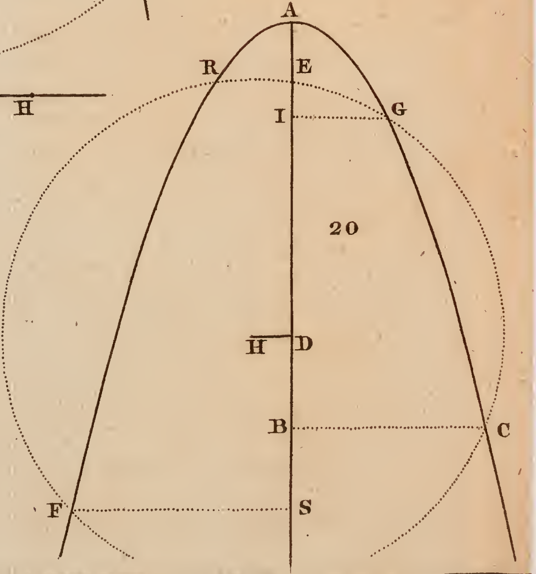
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Pl. III. *pa*

if it is affirmative. But if any of theſe quantities be negative, they muſt be taken the contrary way, to what is directed above. Fig. 21.

3. Then with the radius or diſtance $\sqrt{HD^2 - s}$, and one foot of the compaſſes in H, move the other foot along with the thread, round in a circle, and the weight F will aſcend and deſcend, as the thread EF moves laterally. Obſerve always, when the knot n falls in the line AB, and mark all theſe points, Q, N, O, R. Then the diſtances of theſe points from C, are the roots of the equation; the affirmative on the right, the negative on the left hand of C; thus RC is an affirmative root, and QC, NC, OC, negative ones.

It is plain, this rule is founded on the laſt. For the moving point of the compaſs is always in the curve of a parabola, when the point n is in the line AB. To prove which, ſuppoſe the parabola ADB, to be deſcribed, whoſe focus is E. Then by the property of the figure, $EL + LR = EP + \frac{1}{2}$ parameter $= EP + Pn$ or $En = ED + DC$. Therefore the circle cuts the parabola in L; and the diſtance of L from DC, that is RC is one root of the equation; and the like for the reſt.

SECT.

SECT X.

Rules and Directions for the inveſtigation and ſolution of Problems.

PROBLEM XCIX.

To find if a queſtion be truly limited.

A Queſtion is ſaid to be truly limited, when it admits but of one ſolution; or at moſt, of as many as is the index of the higheſt power of the unknown quantity in the final equation. And whether a queſtion be limited or not, may be known from the equations, by this

RULE.

When the number of unknown quantities, is juſt as many as the number of given equations, not depending on one another; then the queſtion is truly limited.

But when the number of unknown quantities exceeds the number of equations given; then the queſtion is unlimited, and capable of innumerable anſwers.

And when the number of unknown quantities is leſs than the number of given equations; then the queſtion is abſurd and impoſſible, except theſe equations be dependent upon one another; in which caſe the dependent ones may be ſtruck out.

Equations are ſaid to be *dependent* on one another, when they may be formed or derived from one another, by any operations, with the help of the known axioms.

For by Cor. 1, 2. Prob. liii. one unknown quantity may be taken away by each equation; ſo that at laſt there will remain but one equation, and one unknown quantity in it; and therefore it is truly limited.

But if there were more unknown quantities than equations, there will remain more unknown quantities than one, in the laſt equation. And then the queſtion is not limited; for all of them, but one, may be taken at pleaſure; and this is the reaſon of queſtions being unlimited.

Laſtly, if there be more equations than unknown quantities, then at laſt there will remain one unknown quantity for ſeveral equations; and then the queſtion is more than limited; and will therefore be impoſſible. For the unknown quantity being exterminated, there will be an equation conſiſting of all known quantities; which muſt be contradictory to one another, except they were ſome way or other depending on one another, ſo as to make an equality.

SCHOLIUM.

As a problem is truly limited, when the number of independent equations, is equal to the number of unknown quantities; ſo likewiſe a problem is truly limited, though there be never ſo many equations, provided all, above that number, are depending upon theſe, and derived from them. This is plain from any algebraic proceſs; for in the operation, all the ſucceeding equations are derived from theſe, firſt given; and all equations ſo derived, make no alterations in the limitation of the problem.

A problem may be impoſſible and more than limited, though the number of equations be leſs than the number of unknown quantities; and that is when the equations are contradictory.

As if $a + e + 2y = b$,

And $2a + 2e + 4y = c$; a, e, y being unknown quantities, and b, c, known ones. Now if it happen that $c = 2b$, the problem is unlimited; but if c is not $= 2b$, then the problem is impoſſible.

And therefore in general, problems are abſurd, when the equations given are derived from abſurd equations, or may be reduced to ſuch: even though the number of equations be equal to or leſs than the number of unknown quantities.

The equations given in a problem, ought to be independent, otherwiſe they will either be conſequential, or contradictory to one another. In the firſt caſe, you will at laſt find ſome quantity equal to itſelf. And in the ſecond caſe you will arrive at ſome abſurdity, where a greater quantity is equal to a leſs. And it often happens, that at the end of an operation, the equations given, are found to be either dependent or inconſiſtent with one another; which at firſt, could not ſo eaſily be diſcovered.

PROBLEM C.

To inveſtigate an algebraic Problem.

RULES.

1. When a queſtion is propoſed to be reſolved algebraically; the firſt thing to be done, is to conſider the nature and circumſtances thereof, to find out what is given therein, and what required. And the nature and tenor thereof being clearly underſtood; reject every condition or circumſtance, which has no neceſſary connection or relation with the thing enquired after. Then give names to all the quantities concerned in the calculation, whether given or ſought; that is, for the ſeveral numbers or quantities, or at leaſt for the principal of them, put ſo many different letters, as directed in the notation,

notation; taking care to make the ſame letter ſtand invariably for the ſame thing, throughout the whole operation.

And in general problems, it will be convenient to make choice of ſuch letters or ſymbols, as may ſome way repreſent to the mind, the things deſigned by them; as *r* for radius, *s* for ſine, *l* for latus rectum, *v* for velocity, *t* for time, *&c.*

And if there be never ſo many quantities of different ſorts, we may repreſent them by any numbers we like; or even all of them by 1, which is the moſt ſimple notation. Thus we may call any degree of motion 1, any degree of velocity 1, and we may put 1 for any quantity of ſpace, time, matter, *&c.* But then we muſt take care to repreſent other quantities of the ſame ſort, by proportional numbers.

We can alſo meaſure any kind of quantity by any other kind of quantity, by taking parts or degrees of one ſort, proportional to the parts or degrees of the other. Thus, quantities of force may be meaſured by right lines proportional to them; bodies or quantities of matter by their weights; velocities by the ſpaces deſcribed in equal times; and all ſorts of quantities or things by numbers.

2. But as that ſolution of a queſtion is reckoned the more artificial, the fewer unknown quantities are aſſumed at firſt. Therefore when the principal quantities are denoted by letters; ſome of the quantities, that may be eaſily derived from the reſt, are left without a name. As when the whole is given and a part, the other part is eaſily had from thence; or the parts being given, you may find the whole. Alſo when two ſides of a right-angled triangle are denoted in algebraic terms, the third ſide is had from theſe, by addition or ſubtraction of ſquares. Likewiſe three terms of a proportion being given, the fourth term is eaſily

derived from theſe three; and in all ſuch like caſes, where the values of ſome are eaſily derived from the reſt. And by this means there will be fewer terms to exterminate.

3. After the deſignation of the quantities, by letters, as aforeſaid; we muſt next abſtract it from words, and tranſlate it out of the Engliſh into the Algebraic language: that is, we muſt denote all the conditions of it, by ſo many algebraic equations; and this is called *ſtating the queſtion.* In order to this, we muſt ſuppoſe the thing done which was required; and then, without making any difference between the known and unknown quantities, aſſume any of them, known or unknown, to begin your computation from; taking ſuch as you think will bring out the ſimpleſt equation, or give the eaſieſt ſolution. And it is beſt to aſſume that quantity to begin with firſt, which is eaſieſt found or brought to an equation. And therefore it is often more convenient, not to begin with that which is directly required, but with ſome other, from which the quantity required may be eaſily had.

From theſe firſt aſſumed quantities, you muſt proceed in a ſynthetic method to find other quantities wanted, and from theſe to find others, *&c.* according as the nature of the queſtion directs, till you get what equations you want. To this purpoſe, you muſt attend ſtrictly to the nature, deſign, and meaning of the queſtion, and ſearch into all the circumſtances of it, and examine into the particular relations of the quantities to one another; ſo that from thence you may get a proper number of equations. But ſometimes theſe equations cannot be had from the words of the queſtion; but depend upon the hidden properties of the quantities concerned therein; and then the equations are to be deduced from them, by a proper chain of reaſoning, according to the nature of the ſubject under conſideration. Thus, in numerical queſtions, we muſt proceed by the properties of numbers: in geometrical problems, by the elements of geometry: in mechanical problems, by the principles of mechanics: in trigonometrical problems, by the rules of trigonometry: in philoſophical problems, by the laws of motion; and ſo of other ſubjects. And here great care muſt be taken that your equations do not depend upon one another; and that there be as many as there are unknown quantities, otherwiſe the queſtion will not be limited.

4. Having got a proper number of equations, our buſineſs is now, to exterminate them one by one, as faſt as we can, till there only remains one unknown quantity, in one final equation: then the problem is ſaid to be *brought to a ſolution.* And by theſe equations, you muſt exterminate theſe quantities firſt, that are moſt eaſily exterminated; that is, the ſimpleſt firſt, and ſo on; till the quantity that remains at laſt, may give the ſimpleſt equation poſſible; or more ſimple than if any other of the unknown quantities remained in the final equation. And in all your proceſs, great care muſt be taken, to keep to a juſt equality; which will certainly be, if you obſerve all along, to work according to theſe juſt rules or axioms, at the beginning of this book.

5. As to the chuſing fit terms or quantities to begin the calculation with; it ſometimes happens that there is ſuch a relation of two terms of the queſtion, when compared with the reſt, that in making uſe of either of them, they will bring out equations exactly alike; or that both, if they are made uſe of together, ſhall bring out the ſame final equation, as to form. Then it will be the beſt way to make uſe of neither of theſe terms; but inſtead

thereof, to chuſe ſome third, which has a like relation to both. As ſuppoſe the half ſum or half difference, or perhaps a mean proportional; or any other quantity related to both indifferently, and without a like.

6. The proper deſignation of the terms will often much abridge the operation. As if two numbers are ſought, whoſe ſum or difference (n) is given, it will be convenient to take $\frac{1}{2}n + a$, and $\frac{1}{2}n - a$, for the numbers.

Alſo when ſeveral numbers are ſought in arithmetical progreſſion, where the common difference (d) is given; we may properly put $a - d$, a, $a + d$, for the numbers, when there are three: or $a - 1\frac{1}{2}d$, $a - \frac{1}{2}d$, $a + \frac{1}{2}d$, $a + 1\frac{1}{2}d$, for the numbers, when four are required; and ſo on.

Again, if ſeveral numbers are ſought in geometrical progreſſion; put aa, ae, ee, for three numbers: and a^3, a^2e, ae^2, e^3, for four numbers: and a^4, a^3e, a^2e^2, ae^3, e^4, for five numbers; and ſo on. Or denote them by ſuch other ſeries, as will give them all, with the feweſt letters.

7. Sometimes problems will run up into very high equations, where the unknown quantities cannot be expunged without great difficulty. Therefore in ſuch a caſe, if you can ſubſtitute new letters for the ſums, or products, or powers, *&c.* of ſome of the old quantities; and then expunge all theſe old ones, and get a proper number of equations; you may often find the value of theſe new quantities, by eaſy and low equations; from whence the old quantities may be more eaſily determined. And you muſt find theſe new quantities by trials, ſuch, that when they are ſubſtituted, they may render the equations eaſier. See Prob. xxiv, xxv. B. II.

Likewiſe in any operation, when you have a multitude of unknown quantities, for the coefficient o any power of the unknown quantity; put a ſingl lette letter for them all, which will much abridge the operation.

8. In geometrical problems, there is often more labour and ſkill required, than in numerical ones. In theſe you muſt firſt draw a figure, according to what the queſtion requires to be done. And then it is often requiſite to produce right lines; or to draw lines parallel or perpendicular to other lines; and to certain points, or through certain points; or to make ſimilar triangles, and ſuch like; all preparatory for the ſolution of the problem: always endeavouring to reſolve the ſcheme into ſimilar triangles, or right-angled ones, or given ones. Then aſſume ſuch a line, *&c.* for your unknown quantity, as you judge will bring out the ſimpleſt equation. For you may begin your computation with any quantity, known or unknown: which done, you muſt proceed ſynthetically to find the reſt. In general, let theſe quantities be denoted by letters, that lie neareſt the given parts of the figure, and by means of which other parts adjoining may be eaſily had, without ſurds. In triangles draw a perpendicular from the end of a given ſide, and oppoſite to a given angle. Such preparations as theſe being made, juſt as you find neceſſary for the method of ſolution you intend to try; purſue your computation according to the nature and property of lines, and the conditions given in the queſtion, proceeding from the quantities aſſumed, to other quantities, as the relation of the lines direct; till you get two values for one and the ſame quantity, or find one quantity denoted two different ways, by which you will get an equation. The general principles for carrying on the computation are ſuch as theſe; the addition or ſubtraction of lines, to find the ſum or difference. The proportionality of lines (ariſing from ſimilar triangles), where three terms being given to find a fourth. The addition or ſubtraction of ſquares in right-angled triangles, where

two ſides being given, the third may be found. Likewiſe the doctrine of proportion will be of frequent uſe. Beſides we muſt make uſe of ſuch propoſitions in geometry as are ſuitable to the purpoſe; ſuch as B. I. prop. 1, 2, 4, 8, 10, 11. B. II. 2, 3, 10, 13, 14, 15, 18, 21, 24, 25, 26. B. III. 1, 6, 7, 17, 20. B. IV. 9, 12, 13, 14, 16, 17, 20, 27, 31; and ſome in the following books, as occaſion requires. By help of theſe principles, and a chain of right reaſoning, we ſhall obtain as many equations as unknown quantities, which being had, we muſt change our method, and exterminate the ſuperfluous quantities, and find the root of the final equation.

9. If the method you go upon at firſt, for the ſolution of the problem, proceeds but badly, as running into high equations and ſurds. You muſt draw freſh ſchemes, and begin your computation anew, till you have hit on a method as elegant as poſſible. For the principal art, of reſolving problems, is to frame the poſitions with ſuch judgment, that the ſolution may end in as ſimple an equation as poſſible. For ſome methods will produce more intricate equations and ſolutions, than others. But the ſkill of finding out the moſt ſimple and eaſy ways of reſolution is not to be preſcribed by any rules, but is only attained by conſtant practice and experience.

10. If you have any doubt what quantity to take for the quantity ſought, ſo as to bring out the ſimpleſt equation. Suppoſe you have got a final equation with x; take ſome other quantity y, which you ſuſpect may be as ſimple, ſeek an equation between x and y; then if y be of as high a power as x; the final equation, if y were uſed, would be as high as is the final equation with x.

Or, Having got an equation between x and y; ſubſtitute for x its value in terms of y, in the ſaid final equation with x; and you will find what power er y will ariſe to, without forming the proceſs anew for y. But if the equation between x and y be not a ſimple equation; it will often be as well to begin the proceſs anew for y.

Or, If there be ſeveral quantities, and you do not know which will bring out the ſimpleſt equation. Put letters for them all, and get as many equations. Then by expunging ſuch as are moſt eaſily expunged; you will, for the moſt part, get the moſt ſimple equation.

11. Laſtly, when the final equation is obtained, extract its root by Sect. VIII, and you have the anſwer in numbers.

Note, The numbers given in a queſtion, cannot always be taken at pleaſure, but muſt often be ſubject to one or more determinations or reſtrictions, which for the moſt part are diſcoverable by the theorem reſulting from the reſolution of the queſtion.

12. When you have an equation containing the quantity ſought; and the equation is alſo effected with a ſecond unknown quantity, which you want to get rid of; the extermination of which runs you to a very high power. Now if it happens that this ſecond unknown quantity, is but in a few of the terms, which are but ſmall in reſpect of the reſt. Then if you can nearly gueſs at its value, you may retain it in the equation, putting that value for it, which will make little difference in the equation, among ſo many quantities, if you miſs its value a little. Then the root of the equation being extracted will give the other unknown quantity very near. And this being had, the ſecond unknown quantity will then be found more exactly, and may be ſubſtituted for it again, and the operation repeated, &c.

And one may often gueſs nearly at the value of this ſecond quantity, from the conditions of the problem; eſpecially if it be a geometrical one, from the conſtruction of the figure.

Theſe ſorts of equations may alſo be reſolved by the Rule of Falſe Poſition, as directed in Pr. xcii. Rule 5; as alſo by Prob. xcv.

13. When you want to compute a problem for ſome practical uſe in common life, but by purſuing it in its mathematical rigour, you fall upon ſome irreſolvable equations or intricate ſurds or ſeries. Then you may often reſolve it on very ſimple principles, by neglecting ſuch quantities or ſuch conditions, as ſerve only to embarraſs the problem, but make little or no difference for the purpoſe you want it. In ſuch caſe, neglect ſuch quantities or ſuch conditions, as are of little moment; and inſtead of ſuch quantities as make the calculation difficult, take others nearly equal to them, which will make the operations more ſimple, or as ſimple as poſſible. Or ſome of the leaſt moment may be entirely left out. And thus one may come at an eaſy ſolution of the problem.

Theſe are the general rules of working; all which will be made clear, by the examples in the following Book II.

BOOK II.

The Solution of Problems.

A *Problem* is a queſtion propoſed to be reſolved; and the *Solution* of a problem, is the finding ſuch numbers, lines, *&c.* as will fulfil the conditions of the queſtion.

Of problems theſe are *determined*, that have a determinate number of anſwers: and *indetermined*, which have innumerable anſwers.

Problems are of ſeveral kinds, as numerical, geometrical, trigonometrical, philoſophical, mechanical, *&c.*

A problem of one, two, three, *&c.* dimenſions, is that which has one, two, three, *&c.* ſolutions or anſwers.

We have hitherto been laying down ſuch rules, as are neceſſary for the inveſtigation and ſolution of problems. The reader muſt take particular care, to make himſelf well acquainted with theſe rules, and keep them in mind, ſo that he may have them ready for uſe, upon all occaſions; for without them no problem can be ſolved. But as precepts are but of little uſe without examples, and generally reach no farther than mere ſpeculation; I ſhall therefore, in the next place, apply them to practice, and that in the ſolution of a great variety of problems, in the moſt material branches of the mathematics; which I ſhall now begin with directly.

SECT. I.

Numerical Problems.

PROB. I.

There are two numbers whoſe ſum is 25, *and the proportion of one to the other is as* 2 *to* 3, *what are the numbers?*

Suppoſe	1	$a =$ greater.
	2	$e =$ leſſer.
per queſt.	3	$e : a :: 2 : 3$
$3 \times$	4	$2a = 3e$
$4 \div (2)$	5	$a = \frac{3}{2}e$
per queſt.	6	$a + e = e + \frac{3}{2}e = 25$
$6 \times (2)$	7	$2e + 3e = 5e = 50$
$7 \div (5)$	8	$e = \frac{50}{5} = 10$, the leſſer.
5,	9	$a = \frac{3}{2}e = 15$, the greater.

Otherwiſe thus,

Suppoſe	1	$a =$ greater, $s =$ ſum, $= 25$; then $s - a =$ leſſer.
per queſt.	2	$2 : 3 :: s - a : a$
$2 \times$	3	$2a = 3s - 3a$
3 tranſ.	4	$5a = 3s$
$4 \div (5)$	5	$a = \frac{3s}{5} = \frac{75}{5} = 15$, the greater number.
	6	$s - a = 10$, the leſſer.

PROB.

PROB. II.

A man having a certain number of pence, gives to A $\frac{1}{2}$ *of them, to* B $\frac{1}{4}$, *to* C $\frac{1}{8}$, *and to* D $\frac{1}{12}$, *and then had* 3 *remaining. How many had he at firſt?*

Let	1	$a =$ number of pence he had.
per queſt.	2	$\frac{1}{2}a + \frac{1}{4}a + \frac{1}{8}a + \frac{1}{12}a = a - 3$
2,	3	$\frac{23}{24}a = a - 3.$
3 tranſ.	4	$\frac{1}{24}a = 3$
4 ×	5	$a = 72$

PROB. III.

A man hired a labourer, on condition, that for every day he wrought, he ſhould have 12d. *and for every day he idled, he ſhould forfeit* 8 d. *After* 390 *days, neither of them was in debt. To find the number of work days and play days.*

Let	1	$a =$ the working days, $b = 390$; then $\underline{b - a} =$ the play days.
per queſt.	2	$12a = 390 - a \times 8 = 3120 - 8a,$
2 tranſp.	3	$20a = 3120,$
3 ÷ (20)	4	$a = \frac{3120}{20} = 156,$ the work days.
1,	5	$b - a = 234,$ the playing days.

PROB IV.

Some young men and maids had a reckoning of 37 *ſhillings*; *and every man was to pay* 3 *ſhillings, and every maid* 2; *now if there had been as many men as maids, and maids as men, the reckoning would have come to* 4 *ſhillings leſs. What is the number of each?*

Suppoſe	1	$a =$ men, $e =$ maids, $b = 37$, $c = 4$.
perqueſt	2	$3a + 2e = b$
	3	$2a + 3e = b - c$
$2 \times (3)$	4	$9a + 6e = 3b$
$3 \times (2)$	5	$4a + 6e = 2b - 2c$
$4 - 5$	6	$5a = b + 2c$
$6 \div (5)$	7	$a = \frac{b + 2c}{5} = 9$, the men.
$2 - 3a$	8	$2e = b - 3a$
$8 \div (2)$	9	$e = \frac{b - 3a}{2} = 5$, the women.

PROB. V.

A man being aſked what a clock it was? anſwered, that it was between 8 *and* 9; *and that the hour and minute hands were both together.*

Let	1	$x =$ time required, $b = 8$, $c = 12$, $d = 1$.
	2	ſince the two hands divide the hour, and the whole circumference in the ſame proportion, therefore $c : x :: d : x - b$,
$2 \times$	3	$cx - cb = dx$
3 tranſ.	4	$cx - dx = cb$
$4 \div$	5	$x = \frac{cb}{c - d} = \frac{96}{11} = \overset{h.}{8}\ \overset{m.}{43}\ \overset{ſec.}{38\tfrac{2}{11}}$

PROB. VI.

A man gives the firſt beggar he meets with, $\frac{1}{6}$ of the pence he had and 4d. *more: to the ſecond $\frac{1}{6}$ the remaining pence and* 8d. *more: to the third $\frac{1}{6}$ the remaining pence and* 12d. *more, and ſo on, increaſing* 4d. *every time, till at laſt he had nothing left; and then all the beggars had equal ſhares. Query, the number of pence and beggars.*

Suppoſe	1	$a =$ pence he had at firſt.
per queſt.	2	$\frac{a}{6} + 4 =$ pence given to the firſt beggar.
1 — 2	3	$\frac{5}{6}a - 4 =$ remainder.
3 ÷ (6)	4	$\frac{5}{36}a - \frac{4}{6} = \frac{1}{6}$ the remainder.
4 + (8)	5	$\frac{5}{36}a - \frac{4}{6} + 8 =$ pence given to ſecond beggar.
2 = 5	6	$\frac{a}{6} + 4 = \frac{5}{36}a - \frac{4}{6} + 8$, *per* queſt.
6 × (36)	7	$6a + 144 = 5a - 24 + 288$
7 tranſ.	8	$a = 120$
2,	9	$20 + 4 = 24 =$ ſhare of each.
8, 9	10	$\frac{120}{24} = 5 =$ number of beggars.

PROB. VII.

There are three numbers, the firſt with $\frac{1}{3}$ *the othe two makes* 14, *the ſecond with* $\frac{1}{4}$ *the other tw makes* 8; *the third with* $\frac{1}{5}$ *the other two make* 8. *What are the numbers?*

Let	1	a, e, y be the numbers.
perqueſt	2	$a + \frac{e+y}{3} = 14$
	3	$e + \frac{a+y}{4} = 8$
	4	$y + \frac{a+e}{5} = 8$
2 × (3)	5	$3a + e + y = 42$
3 × (4)	6	$4e + a + y = 32$
5 — 6	7	$2a - 3e = 10$
5 × (5)	8	$15a + 5e + 5y = 210$
4 × (5)	9	$a + e + 5y = 40$
8 — 9	10	$14a + 4e = 170$
7 × (4)	11	$8a - 12e = 40$
10 × (3)	12	$42a + 12e = 510$
11 + 12	13	$50a = 550$
13 ÷ (5)	14	$a = 11$
7 tranſ.	15	$3e = 2a - 10$
15 ÷ (3)	16	$e = \frac{2a - 10}{3} = 4$
5 tranſp.	17	$y = 42 - 3a - e = 5.$

PROB. VIII.

Having given the ſum of two numbers 8, *and th difference of their ſquares,* 16; *to find the numbers.*

Let	1	x = leſſer number, $a = 8$, $b = 16$
	2	$a - x$ = greater number.

1 @ 2	3	$xx =$ ſquare of the leſſer.
2 @ 2	4	$aa - 2ax + xx =$ ſquare of the greater
4 — 3	5	$aa - 2ax = b$ *per* queſt.
5 tranſ.	6	$2ax = aa - b$
6 ÷ $2a$	7	$x = \frac{aa - b}{2a} = 3$, the leſſer.
2,	8	$a - x = 5$, the greater.

PROB. IX.

There are three numbers, the ſum of the firſt and ſecond is 9, *of the firſt and third* 10, *of the ſecond and third* 13. *What are the numbers?*

Let	1	x, y, z, be the numbers, $a = 9$, $b = 10$, $c = 13$;
*er*queſt	2	$x + y = a$
	3	$x + z = b$
	4	$y + z = c$
2 — x	5	$y = a - x$
3 — x	6	$z = b - x$
4, 5, 6	7	$a - x + b - x = c$
7 tranſ.	8	$2x = a + b - c$
8 ÷ (2)	9	$x = \frac{a + b - c}{2} = 3$
5,	10	$y = a - x = 6$
6,	11	$z = b - x = 7$

PROB. X.

wo travellers A *and* B, 360 *miles diſtance, ſet out at the ſame time.* A *travels* 10 *miles an hour;* B 8. *How far does each travel before they meet?*

Suppoſe	1	A travels x miles, then B travels $360 - x$.
y propor.	2	$x : 360 - x :: 10 : 8$
2 ×	3	$8x = 3600 - 10x$

3 tranſ. | 4 | $18x = 3600$

4 ÷ (18) | 5 | $x = \frac{3600}{18} = 200$, A's journey.

1, | 6 | $360 - x = 160 =$ B's journey.

PROB. XI.

If three agents A, B, C, *can produce the effects a, b, c, in the times e, f, g, reſpectively. In what time will they all jointly produce the effect d?*

Let | 1 | $x =$ time ſought.

by proportion | 2 | $e : x :: a : \frac{ax}{e}$, A's effect in the time x.

3 | $f : x :: b : \frac{bx}{f} =$ B's effect in time x.

4 | $g : x :: c : \frac{cx}{g} =$ C's effect in time x.

2, 3, 4, | 5 | $\frac{ax}{e} + \frac{bx}{f} + \frac{cx}{g} = d$.

5 reduced | 6 | $x = \frac{d}{\frac{a}{e} + \frac{b}{f} + \frac{c}{g}} =$ time required.

PROB. XII.

A woman can buy apples at 10 *a penny, and pears at* 25 *for* 2 *pence; if ſhe lay out* $9\frac{1}{2}$ *pence for* 100 *apples and pears together. How many of each muſt ſhe have?*

Let | 1 | $a =$ apples, then $100 - a =$ pears.

by proportion | 2 | $10 : 1d. :: a : \frac{a}{10}$, price of the apples.

3 | $25 : 2d. :: 100 - a : \frac{200 - 2a}{25}$, price of the pears.

per

per queſt	4	$\frac{a}{10} + \frac{200 - 2a}{25} = 9\frac{1}{2}$.
4 × (50)	5	$5a + 400 - 4a = 475$
5 tranſp.	6	$a = 75$, the apples.
	7	$100 - a = 25$ the pears.

PROB. XIII.

A vintner would mix wine at 10d. *the quart, with another ſort at* 6d; *to make a* 100 *quarts to be ſold at* 7d. *How much of each muſt he take?*

Let	1	$a =$ quarts of 10 penny, $e =$ quarts of 6 penny, $b = 10$, $c = 6$, $m = 100$, $f = 7$.
by proportion	2	$1 : b :: a : ba$, value of a quarts.
	3	$1 : c :: e : ce$, value of e quarts.
per qu.	4	$ba + ce = mf$.
	5	$a + e = m$
$5 - a$	6	$e = m - a$
4, 6	7	$ba + cm - ca = mf$
7 tr.	8	$ba - ca = fm - cm$
8 ÷	9	$a = \frac{fm - cm}{b - c}$
6,	10	$e = \frac{bm - fm}{b - c}$.

PROB. XIV.

A factor exchanged 6 French *crowns and* 2 *dollars for* 45 *ſhillings; and* 9 French *crowns and* 5 *dollars for* 76 *ſhillings. What is the value of a* French *crown and a dollar?*

Suppoſe	1	$x =$ *French* crowns, $y =$ dollars, $a=6$, $b=2$, $d=9$, $e=5$, $c=45$, $f=76$.

per

per qu.	2	$ax + by = c$
	3	$dx + ey = f$
$2 \times e$	4	$eax + eby = ec$
$3 \times b$	5	$bdx + eby = bf$
$4 - 5$	6	$aex - bdx = ec - bf$
$6 \div$	7	$x = \frac{ec - bf}{ae - bd} = 6\frac{1}{12}$.
2,	8	$\frac{ec - bf}{ae - bd} a + by = c$
8 tr.	9	$by = c - \frac{ec - bf}{ae - bd} a = \frac{bfa - bdc}{ae - bd}$
	10	$y = \frac{af - cd}{ae - bd} = 4\frac{1}{4}$.

PROB. XV.

To divide a number b, into four parts; ſo that the firſt being increaſed with d, the ſecond diminiſhed by d, the third multiplied by d, and the fourth divided by d; may be all equal.

Let	1	$a, e, u, y,$ be the four parts.
per qu.	2	$a + e + u + y = b,$
	3	$a + d = e - d$
	4	$a + d = dy$
	5	$a + d = \frac{u}{d}$.
$3 + d$	6	$e = a + 2d$
$4 \div d$	7	$y = \frac{a + d}{d}$
$5 \times d$	8	$u = ad + dd$
2, 6, 7, 8,	9	$a + a + 2d + \frac{a + d}{d} + ad + dd = b$
9 reduced	10	$a = \frac{bd - d^3 - 2dd - d}{dd + 2d + 1}$

6	11	$e = \frac{bd + d^3 + 2dd + d}{dd + 2d + 1}$
7	12	$y = \frac{b}{ad + 2d + 1}$
8	13	$u = \frac{bdd}{dd + 2d + 1}$

PROB. XVI.

A merchant bought (a) buſhels of wheat, (b) buſhels of barley, and (c) buſhels of oats for (m) pounds. Afterwards he bought (d) buſhels of wheat, (e) buſhels of barley, and (f) buſhels of oats, for (n) pounds. And after that, (g) buſhels of wheat, (h) buſhels of barley, and (k) buſhels of oats for (p) pounds, each ſort at one price. What was each per buſhel?

Let	1	x, y, z, be the prices of the wheat, barley, and oats.
	2	$ax + by + cz = m$
per qu.	3	$dx + ey + fz = n$
	4	$gx + hy + kz = p.$
$2 \times fk$	5	$afkx + bfky + cfkz = fkm$
$3 \times ck$	6	$ckdx + ckey + cfkz = ckn$
$4 \times cf$	7	$cfgx + cfhy + cfkz = cfp$
$5 - 6$	8	$afkx - ckdx + bfky - ckey = fkm - ckn.$
$6 - 7$	9	$ckdx - cfgx + ckey - cfhy = ckn - cfp.$
ſubſtit. 8,	10	$Ax + By = C$
9,	11	$Fx + Gy = H.$
10 reduc.	12	$y = \frac{C - Ax}{B}$
11 reduc.	13	$y = \frac{H - Fx}{G}.$
$12 = 13$	14	$\frac{C - Ax}{B} = \frac{H - Fx}{G}.$
$14 \times$	15	$GC - GAx = BH - BFx$

15 tr. 16 $BFx - GAx = BH - GC$

17 $x = \frac{BH - GC}{BE - GA}$.

by reſtitution 10, 11. 18 $x = \frac{bcfkkn + ccfkep + cffkhm - cffbpk - cfkkem - ccfkhn}{bcfkkd + ccfkge + acffkh - bcffgk - acfkke - ccfkhd}$

$18 \div \frac{cfk}{cfk}$ 19 $x = \frac{bkn - bfp + cep - kem + fhm - chn}{bkd - bfg + eeg - aek + afh - cdh}$

x being had, y may be found by ſtep 12th; and then z, by reducing the equation in ſtep 2d.

PROB. XVII.

If the number of oxen a, eat up the paſture b, in the time c; and the oxen d eat up as good a paſture e, in the time f; and the graſs grows uniformly. To find how many oxen will eat up the like paſture g, in the time h.

State it thus:	*Oxen.*	*Weeks.*	*Acres.*
	a	c	b
	d	f	e
	y	h	g

Let 1 y = number of oxen ſought.

2 x = graſs upon an acre at firſt.

3 z = graſs growing upon an acre, in a week afterwards.

4 1 = graſs which an ox eats in a week.

by proportion 5 bx, ex, gx = graſs on the paſtures b, e, g.

6 cbz, fez, hgz = graſs grown afterwards on the paſtures b, e, g, in the times c, f, h.

7 ac, df, hy = graſs eaten by the oxen, a, d, y, in the times c, f, h.

per

per qu.	8	$ac = bx + cbz$
	9	$df = ex + fez$
	10	$yh = gx + hgz$
$8 \times ef$	11	$acef = efbx + efcbz$
$9 \times cb$	12	$cbdf = cbex + cbefz$
$11 - 12$	13	$acef - cbdf = efbx - cbex$,
$13 \div$	14	$x = \frac{acef - cbdf}{efb - ecb}$.
$9 \times g$	15	$gdf = gex + gfez$
$10 \times e$	16	$eyh = egx + ehgz$
$16 - 15$	17	$eyh - gdf = ehgz - gfez$
$17 \div$	18	$z = \frac{eyh - gdf}{ehg - efg}$
$8 \div b$	19	$\frac{ac}{b} = x + cz$
19, 14, 18	20	$\frac{ac}{b} = \frac{acef - cbdf}{efb - ecb} + \frac{ceyh - cgdf}{ehg - efg}$.
Put	21	$p = f - c,\ r = h - f,\ r + p = h - c$.
20, 21	22	$ac = \frac{acef - cbdf}{pe} + \frac{ceyh - cgdf}{reg}\, b$.
$22 \times preg$	23	$acpreg = acefrg - cbdfrg + pceyhb -$ $pcdfgb$
23 tr.	24	$peyhb = apreg - aefrg + bdfrg +$ $pdfgb$.
24,	25	$pebhy = areg \times \overline{p - f} + bdfg \times \overline{r + p}$.
21, 25	26	$pebhy = - aregc + bdfg \times \overline{r + p}$.
26, 21,	27	$pebhy = aceg \times \overline{f - h} + bdfg \times \overline{h - c}$
$27 \div$	28	$y = \frac{aceg \times \overline{f - h} + bdfg \times \overline{h - c}}{beh \times \overline{f - c}}$.

PROB. XVII.

To divide ten thouſand into two ſuch parts, that when each of them is divided by the other, the ſum of the quotients, may be 5.

Let	1	a, e be the parts, $b = 10000$; $c = 5$.
per qu.	2	$a + e = b$
	3	$\frac{a}{e} + \frac{e}{a} = c$
$2 - a$	4	$e = b - a.$
4 ☉ 2	5	$ee = bb - 2ba + aa$
$3 \times ae$	6	$cae = aa + ee$
4, 5, 6	7	$cab - caa = 2aa - 2ba + bb$
$7 \pm$	8	$\overline{2+c}.aa - \overline{2b+bc}.a = -bb.$
$8 \div$	9	$aa - ba = \frac{-bb}{2+c}$
9 reduc.	10	$a = \sqrt{\frac{bb}{4} - \frac{bb}{c+2}} + \frac{b}{2} = 8273,268$
4,	11	$e = 1726,732$

PROB. XVIII.

A general would range his army in a ſquare battle, but finds he has 284 ſoldiers to ſpare; but if he increaſes the ſide of the ſquare with one man, he wants 25 to fill up the ſquare. How many ſoldiers had he?

Let	1	$a =$ ſide of the ſquare.
per qu.	2	$aa + 284 =$ number of men
	3	$\overline{a+1}\vert^{2} - 25 =$ number of men
$2 = 3$	4	$aa + 284 = aa + 2a + 1 - 25$
4 tran	5	$2a = 308$
$5 \div 2$	6	$a = 154$
	7	$aa + 284 = 24000$, the number of men.

PROB. XIX.

Several perſons dining at an inn, the reckoning came to 75 ſhillings; but two of them ſlinking away, the reſt had 10 ſhillings a-piece more to pay. Query, the number of perſons?

Put	1	$a =$ number of perſons at firſt.
per qu.	2	$\frac{175}{a} =$ each man's ſhot.
	3	$\overline{a-2} \times \overline{\frac{175}{a}+10} = 175.$
3 ×	4	$175a + 10aa - 350 - 20a = 175a$
4 tranſ.	5	$10aa - 20a = 350$
5 ÷	6	$aa - 2a = 35$
6 reſolved	7	$a = 7$
2	8	$\frac{175}{a} = 25$, the club.

PROB. XX.

There is a number, conſiſting of three digits, whoſe product is 315, and the ſum of the firſt and laſt is double the ſecond; and that number with 396 added makes a number conſiſting of the ſame digits but inverted.

Suppoſe	1	a, e, y the digits. $\quad \begin{matrix} a & e & y \\ y & e & a \end{matrix}$
per qu.	2	$a + y = 2e$
	3	$aey = 315$
	4	$100a + 10e + y + 396 = 100y + 10e + a,$
4 ±	5	$99a + 396 = 99y$
5 ÷ (99)	6	$a + 4 = y$
2 — a	7	$2e - a = y$
6 = 7	8	$a + 4 = 2e - a$

8,	9	$a = e - 2$
6,	10	$y = e + a$
3, 9, 10	11	$aey = e \times \overline{e - 2} \times \overline{e + 2} = e^3 - 4e = 315$
11 reſolv.	12	$e = 7$
9, 10	13	$a = 5,\ y = 9.$

PROB. XXI.

To find the value of a, when $\sqrt{a^3} - \sqrt[3]{aa} = 4.962a.$

Put	1	$b = 4.962$
per queſt.	2	$a^{\frac{3}{2}} - a^{\frac{2}{3}} = ba$
2,	3	$a^{\frac{9}{6}} - a^{\frac{4}{6}} = ba$
aſſume	4	$x^6 = a$, then
4,	5	$a^{\frac{9}{6}} = x^9$, and $a^{\frac{4}{6}} = x^4$
3, 4, 5	6	$x^9 + x^4 = bx^6$
$6 \div x^4$	7	$x^5 - 1 = bxx$
7 tranſ.	8	$x^5 - bxx = 1$
8 reſolv.	9	$x = 1.74256$
4,	10	$a = 27.998$

PROB. XXII.

Given {	1	$a^3e^3 + a^2e^3 = 234000 = b.$
	2	$ae + ae^2 + ae^3 = 1860 = c.$
		to find a, e.
$2 \div$	3	$a = \frac{c}{e + ee + e^3}$
$1 \div e^3$	4	$a^3 + a^2 = \frac{b}{e^3}$
3, 4	5	$\frac{c^3}{\overline{e + ee + e^3}\vert^3} + \frac{cc}{\overline{e + ee + e^3}\vert^2} = \frac{b}{e^3}.$

5 × e^3	6	$\frac{c^3}{\overline{e+e+ee}\vert^3} + \frac{cce}{\overline{1+e+ee}\vert^2} = b$
6 ×	7	$b \times \overline{1+e+ee}\vert^3 = cce \times \overline{1+e+ee} + c^3$
reduced	8	$be^6 + 3be^5 + 6be^4 + \begin{matrix}7be^3\\-c^2e^3\end{matrix} + \begin{matrix}6be^2\\-c^2e^2\end{matrix} \begin{matrix}+3be\\-c^2e\end{matrix} + b \Big\} = c^3$

PROB. XXIII.

There is a caſk of rum, out of which was taken 41 gallons, and filled up with water, and the ſame repeated three times more. At laſt there was found by the proof, to remain 25.2935 gallons of rum in it. What was the content of the caſk?

Let	1	$b=41$, $c=25.2935$, $a=$ caſk's content, then $a-b=$ firſt remainder.
	2	And ſince the quantity of liquor is as the ſpace it poſſeſſes; therefore $a : a-b$ (1 rem.) $:: a-b : \frac{\overline{a-b}\vert^2}{a}$ (2 rem.) $:: \frac{\overline{a-b}\vert^2}{a} : \frac{\overline{a-b}\vert^3}{aa}$ (3 rem.) $:: \frac{\overline{a-b}\vert^3}{aa} : \frac{\overline{a-b}\vert^4}{a^3}$ (4 rem.)
per queſt.	3	$\frac{\overline{a-b}\vert^4}{a^3} = c.$
3 ©	4	$a^4 - 4ba^3 + 6bba^2 - 4b^3a + b^4 = ca^3$
4 tr.	5	$a^4 \begin{matrix}-4b\\-c\end{matrix} a^3 + 6bba^2 - 4b^3a = -b^4$
5 reſol.	6	$a = 124.84$ gallons.

PROB. XXIV.

Given	1	$x^3+x^2y+y^2x+y^3=dxy.$
	2	$x^6+x^4y^2+y^4x^2+y^6=bbxxyy.$
1,	3	$\overline{x^2+y^2}\times\overline{x+y}=dxy$
2,	4	$\overline{x^4+y^4}\times\overline{x^2+y^2}=bbxxyy$
Put $\left\{\begin{matrix}\\ \\ \end{matrix}\right.$	5	$xy=a$
	6	$x+y=e$
3, 5, 6	7	$\overline{ee-2a}\times e=da=\overline{x^2+y^2}\times\overline{x+y}$
$7\div e$	8	$ee-2a=\frac{da}{e}=xx+yy$
4, 8	9	$\frac{ddaa}{ee}-2aa=x^4+y^4$
$9\times\frac{da}{e}$	10	$\overline{\frac{ddaa}{ee}-2aa}\times\frac{da}{e}=\overline{x^4+y^4}\times$ $\overline{x^2+y^2}=bbxxyy=bbaa.$
$10\times$	11	$\overline{dd-2ee}\times da=bbe^3$
$11\div$	12	$a=\frac{bbe^3}{d^3-2dee}$
8, 12	13	$e^3-\frac{2bbe^4}{d^3-2dee}=\frac{db^2e^3}{d^3-2dee}$
$13\times$	14	$d^3e^3-2de^5-2bbe^4=dbbe^3$
$14\div$	15	$d^3-2dee-2bbe=dbb$
		$ee+\frac{bb}{d}e=\frac{dd-bb}{2}$, a quadratic which gives *e*, and then *a* by ſtep 12, and *x* and *y* may be found from ſtep 5 and 6.

PROB.

PROB. XXV.

Given	1	$a^5e - bda^4 + 2ba^3ee + bbae^3 - 2dbba^2e - b^3de^2 = d^6$
	2	$a^4ee - 2bda^3e + bbdda^2 + ba^2e^3 - 2dbbae^2 + bbd + b^3d^2e = b^4d^2$.
that is, 1, 2	3	$\overline{aa+be}\big\vert^2 \times \overline{ae-bd} = d^6$.
	4	$\overline{ae-bd}\big\vert^2 \times \overline{aa+be} = b^4d^2$
Put	5	$aa + be = x$
	6	$ae - bd = y$
3, 5, 6	7	$x^2y = d^6$
4, 5, 6	8	$y^2x = b^4dd$.
7 × 8	9	$x^3y^3 = b^4d^8$.
9 ш 3	10	$xy = bdd\sqrt[3]{bdd}$
7 ÷ 10,	11	$x = \dfrac{d^6}{bdd\sqrt[3]{bdd}} = \dfrac{d^3}{bb}\sqrt[3]{bbd}$
8 ÷ 10	12	$y = \dfrac{b^4dd}{bdd\sqrt[3]{bdd}} = \dfrac{bb}{d}\sqrt[3]{bbd}$
5 − be	13	$aa = x - be$
6, +	14	$a = \dfrac{y + bd}{e}$
13, 14	15	$aa = \dfrac{yy + 2bdy + bbdd}{ee} = x - be$
15 red.	16	$xee - be^3 = yy + 2bdy + bbdd$, a cubic equation which gives e, whence a is known by ſtep 14.

PROB. XXVI.

To find four numbers x, y, z, v, having the product of every three given.

Suppoſe	1	$xyz = b$
	2	$yzv = c$
	3	$zvx = d$
	4	$vxy = f$

1 × 2 × 3 × 4

1×2×3×4	5	$x^3y^3z^3v^3 = bcdf$
5 lw 3	6	$xyzv = \sqrt[3]{bcdf}$
6 ÷ 2	7	$x = \frac{\sqrt[3]{bcdf}}{c}$
6 ÷ 3	8	$y = \frac{\sqrt[3]{bcdf}}{d}$
6 ÷ 4	9	$z = \frac{\sqrt[3]{bcdf}}{f}$
6 ÷ 1	10	$v = \frac{\sqrt[3]{bcdf}}{b}$

PROB. XXVII.

To find 3 numbers, x, y, z, having the product of each and the sum of the other two, given.

per qu.	1	$x \times \overline{y + z} = b$
	2	$y \times \overline{x + z} = c$
	3	$z \times \overline{x + y} = d$
1+2+3	4	$2xy + 2xz + 2yz = b + c + d$
4 ÷ (2)	5	$xy+xz+yz = \frac{b+c+d}{2} = s$, by ſubſt.
5 − 1	6	$yz = s - b$
5 − 2	7	$xz = s - c$
5 − 3	8	$xy = s - d$
6×7×8	9	$xxyyzz = \overline{s-b} \times \overline{s-c} \times \overline{s-d}$
9 lw 2	10	$xyz = \sqrt{\overline{s-b} \times \overline{s-c} \times \overline{s-d}}$
10 ÷ 6	11	$x = \sqrt{\frac{\overline{s-c} \times \overline{s-d}}{s-b}}$.
10 ÷ 7	12	$y = \sqrt{\frac{\overline{s-b} \times \overline{s-d}}{s-c}}$.
10 ÷ 8	13	$z = \sqrt{\frac{\overline{s-b} \times \overline{s-c}}{s-d}}$.

PROB. XXVIII.

To find any polygonal or figurate number.

A figurate or poligonal number is the ſum of a ſeries of numbers in arithmetical progreſſion from 1. And theſe are ſo called, becauſe they denote the number of points, which fill a regular poligon, placed at equal diſtances, on lines drawn parallel and equidiſtant, to the ſides of the figure. The following table ſhews the arithmetic proportionals, and the poligonal numbers formed from them. The numbers of the arithmetical ſeries ſhew what number of points are placed on the ſeveral parallel lines of the poligon; and the poligonal numbers, ſhew the whole number of the points contained in the figure. 22.

Rank.	Arithm. proportionals.	Poligonal numbers.	Names.
1	1, 1, 1, 1, 1, 1	1, 2, 3, 4, 5, 6	laterals.
2	1, 2, 3, 4, 5, 6	1, 3, 6, 10, 15, 21	triangul.
3	1, 3, 5, 7, 9, 11	1, 4, 9, 16, 25, 36	quadran.
4	1, 4, 7, 10, 13, 16	1, 5, 12, 22, 35, 51	pentang.
5	1, 5, 9, 13, 17, 21	1, 6, 15, 28, 45, 66	hexang.
6	1, 6, 11, 16, 21, 26	1, 7, 18, 34, 55, 81	heptang.
7	1, 7, 13, 19.25, 31	1, 8, 21, 40, 65, 96	Octang.

Let 1 $r =$ any rank, $x =$ poligonal number ſought.
$n =$ place of x; then $r - 1 =$ common diff. of the arithm. ſeries.

arith. prop Pr. 6. 2 $1 + \overline{n - 1} \times \overline{r - 1} =$ the n^{th} term in the arithmetic progreſſion.

ib. Pr. 7. 3 $\frac{2 + \overline{n - 1} \times \overline{r - 1}}{2} \times n = n^{th}$ term in the poligonal numbers.

1, 3, 4 $x = \frac{2 + \overline{n - 1} \times \overline{r - 1}}{2} \times n.$

SECT. II.

Of Intereſt and Annuities.

PROB. XXIX.

The principal, time, and rate of intereſt being given; to find the amount, or money due at the end of that time; at ſimple intereſt.

Let	1	$p =$ principal, $t =$ time, $r =$ rate of intereſt of 1 *l.* for a certain time, as a year, *&c.* $s =$ ſum of all the arrears.
by proportion	2	$1 : r :: p : rp$, the intereſt of p for a year.
	3	$1 : rp :: t : prt$, the intereſt for the time t.
	4	$p + prt =$ whole arrear at the end of the time t.
1, 4	5	$p + prt = s$, the arrear ſought.

Cor. 1. *Hence* $p = \dfrac{s}{rt + 1}$, *when* s, r, t, *are given.*

Cor. 2. $t = \dfrac{s - p}{pr}$, *when* s, p, r, *are given.*

Cor. 3. $r = \dfrac{s - p}{pt}$, *when* s, p, t, *are given.*

PROB.

PROB. XXX.

The annuity, time and rate of intereſt being given; to find the arrear, at the end of that time, at ſimple intereſt.

Put	1	$a =$ annuity or yearly rent; $t =$ time of forbearance; $r =$ intereſt of 1 *l.* for a year, *&c.* $s =$ whole arrear.
by proportion	2	$0 =$ intereſt due at one year's end.
	3	$ra =$ intereſt at 2 year's end.
	4	$2ra =$ intereſt at 3 year's end.
	5	$3ra =$ intereſt for 4 years.
	6	$\overline{t-1}\,.\,ra =$ intereſt for t years.
	7	$ta =$ rents due at the end of t years.
2, 3, 4, 5, 6, 7	8	$0+1+2+3\ldots$ to $t-1 \times$ into $ra+ta=s$.
arith. prop Prop. 7.	9	$0+1+2+3\ldots t-1=\frac{t\times\overline{t-1}}{2}$.
8, 9,	10	$\frac{t\times\overline{t-1}}{2}ra+ta=s$.
10,	11	$\frac{\overline{t-1}\,.\,r+2}{2}ta=s$.

Cor. 1. $a=\frac{2s}{\overline{\overline{t-1}\,.\,r+2}\times t}$

Cor. 2. $t=\sqrt{\frac{2s}{ar}+\overline{\frac{2-r}{2r}}\Big|^2}-\frac{2-r}{2r}$

Cor. 3. $r=\frac{2s-ta}{\overline{t-1}\times ta}$

PROB.

PROB. XXXI.

To find the preſent worth of an annuity, to continue a given time, at a given rate of ſimple intereſt.

Let	1	p = preſent worth, a = annuity, t = time, r = intereſt of 1 *l.*
Prob. 29.	2	$p+prt=s$
Prob. 30.	3	$\frac{\overline{t-1}.r+2}{2}ta=s$
2 = 3	4	$p+prt=\frac{\overline{t-1}.r+2}{2}ta$
4 ÷	5	$p=\frac{\overline{t-1}.r+2}{2rt+2}ta=\frac{\frac{t-1}{2}r+1}{rt+1}ta.$

Cor. 1. $a=\frac{rt+1}{\frac{t-1}{2}r+1}\times\frac{p}{t}.$

Cor. 2. $tt+\overline{\frac{2}{r}-\frac{2p}{a}-1}\times t=\frac{2p}{ra}$, *whence t may be found.*

Cor. 3. $r=\frac{2ta-2p}{\overline{2p-\overline{t-1}.a}\times t}.$

PROB. XXXII.

The principal, time, and rate of intereſt being given; to find the amount at the end of that time, at compound intereſt.

Let	1	p = principal, t = time, r = intereſt of 1 *l.* R = 1 + r the amount of 1 *l.* and its intereſt. s = ſum of money due at the end of that time.

per

per queſt.	2	$1 + r$ or $R =$ money due at 1 year's end.
by proportion	3	$1 : R :: R : RR =$ money due at 2 year's end.
	4	$1 : R :: RR : R^3 =$ money due at 3 year's end.
	5	$R^t =$ money due at t year's end.
	6	$1 : R^t :: p : pR^t =$ the amount of p for the time t.
1, 6	7	$pR^t = s$.

Cor. 1. $p = \frac{s}{R^t}$.

Cor. 2. $R^t = \frac{s}{p}$, or $t = \frac{\log: s - \log: p}{\log: R}$.

Cor. 3. $R = \sqrt[t]{\frac{s}{p}}$, or $\log: R = \frac{\log: s - \log: p}{t}$.

PROB. XXXIII.

The annuity, time, and rate of intereſt, being given; to find the arrears due at the end of that time, at compound intereſt.

Let	1	$a =$ annuity, or yearly rent, $t =$ time of forbearance, $r =$ intereſt of 1 *l.* for a year, *&c.* $R = 1 + r$, $s =$ ſum of all the arrears.
	2	$a =$ money due at 1 year's end.
by Prob. 32.	3	$2a + ra = a + Ra =$ arrear at 2 years end.
	4	$a + aR + aRR =$ arrear in three years.
	5	$a + aR + aR^2 + aR^3 =$ arrear for 4 years.
	6	$a + aR + aR^2 + aR^3 \ldots\ldots$ to $aR^{t-1} =$ arrear for t years.

geom.

geom. propor. prop. 26	7	$1 + R + R^2 + R^3 \ldots$ to $R^{t-1} = \frac{R \times \overline{R^{t-1} - 1}}{R - 1} = \frac{R^t - 1}{r}$.
6, 7	8	$\frac{R^t - 1}{r} a =$ money due at the end of t years.
1, 8,	9	$\frac{R^t - 1}{r} a = s$.

Cor. 1. $a = \frac{rs}{R^t - 1}$.

Cor. 2. $R^t = \frac{rs}{a} + 1$, or $t = \frac{\log: \overline{\frac{rs}{a} + 1}}{\log: R}$.

Cor. 3. $\frac{s}{a} R - R^t = \frac{s - a}{a}$, *whence* R *may be found, and then r.*

PROB. XXXIV.

To find the preſent worth of an annuity, to continue a given time, at a given rate of compound intereſt.

Let	1	$p =$ preſent worth, $a =$ the annuity, $t =$ the time, $r =$ intereſt of 1 *l.* $R = 1 + r$.
Prob. 32.	2	$pR^t = s$.
Prob. 33.	3	$\frac{R^t - 1}{r} a = s$.
2 = 3	4	$pR^t = \frac{R_t - 1}{r} a$
4 ÷ R^t	5	$p = \frac{R^t - 1}{rR^t} a = \frac{1 - \frac{1}{R^t}}{r} a$.

Cor. 1. $a = \dfrac{pr}{1 - \dfrac{1}{R^t}}$.

Cor. 2. $R^t = \dfrac{a}{a - pr}$, or $t = \dfrac{\log:a - \log:\overline{a - pr}}{\log:R}$.

Cor. 3. $R^t + \dfrac{a}{p} R^t - R^{t+1} = \dfrac{a}{p}$, *whence* R *and* r *will be found.*

PROB. XXXV.

To find the value of an annuity to continue for ever, at a given rate of compound intereſt.

Let	1	p = preſent worth, a = annuity, r = intereſt of 1 *l.* $R = 1 + r$.
Pr. 34. ſtep. 5.	2	$p = \dfrac{R^t - 1}{rR^t} a$
Prob. 73. cor. 7.	3	but ſince t is infinite, R^t is infinitely greater than 1, whence $R^t - 1 = R^t$.
2, 3,	4	$p = \left(\dfrac{R^t}{rR^t} a =\right) \dfrac{a}{r}$.

Cor. 1. $a = pr$.

Cor. 2. $r = \dfrac{a}{p}$.

PROB. XXXVI.

At what rate of intereſt will 100l. *amount to* 200l. *in* $9\frac{3}{4}$ *years, at compound intereſt.*

Let	1	r = rate of 1 *l.* $R = 1 + r$, $t = 9\frac{3}{4}$.
Prob. 32,	2	$100R^{\frac{39}{4}} = 200$. *per* queſt.

2 ÷	3	$R^{\frac{39}{4}} = 2$
3 ⊚ 4	4	$R^{39} = 16$
4 ɯ 39	5	$R = \sqrt[39]{16} = 1.0737$ by logarithms
5 — 1	6	$R - 1 = r = .0737.$
6 × 100	7	$7.37 =$ rate of intereſt *per cent.*

PROB. XXXVII.

If a principal x be put out at compound intereſt, for x years, at x per cent. *to find the time x, in which it will gain x.*

Prob. 32.	1	$pR^t = s$
per queſt.	2	$p = x$, $r = \frac{x}{100}$, $R = 1 + \frac{x}{100}$, $t = x$ $s = 2x.$
1, 2	3	$x \times \overline{1 + \frac{x}{100}}\Big\vert^{x} = 2x.$
3 ÷ x	4	$\overline{1 + \frac{x}{100}}\Big\vert^{x} = 2.$
nature of logs.	5	$x \times \overline{1 + \frac{x}{100}} \times M = .3010300$
Prob. 84. cor. 1.	6	$x \times : \frac{x}{100} - \frac{xx}{20000} + \frac{x^3}{3000000}$ &c. $: = \frac{.30103}{M}$
6,	7	$\frac{xx}{100} - \frac{x^3}{20000} + \frac{x^4}{3000000}$ &c $= ,693147$
by reverſ.	8	$x = 8.49824$ years.

PROB. XXXVIII.

Given the rate per cent. *for a year* (5 l.), *to find what the amount of any ſum* (100 l.), *will be at the year's end, at compound intereſt; ſuppoſing it to ariſe from the principal and intereſt due every day*, &c.

Let { 1 $r =$ intereſt of 1 *l.* for a year.

2 $n = 365$ the parts of a year.

3 $\frac{r}{n} =$ intereſt for 1 day.

3, 4 $1 + \frac{r}{n} =$ money due at one day's end.

Prob. 32. 5 $\overline{1 + \frac{r}{n}}\Big|^{n} =$ money due at the year's end.

by logs. 6 $n \times \log : \overline{1 + \frac{r}{n}} = \log :$ amount for a year $= .0215694$.

6, 7 $1.0509 =$ amount for a year.

6×100 8 $105.09 =$ amount of 100 *l.*

or 5, 9 $\overline{1 + \frac{r}{n}}\Big|^{n} = 1 + r + \frac{n.\overline{n-1}}{2nn}rr + \frac{n.\overline{n-1}.\overline{n-2}}{2.3n^{3}}r^{3}$ &c. the amount for a year.

10 If the intereſt is ſuppoſed to gain intereſt every moment, by becoming part of the principal; then n is infinite, and $\overline{1 + \frac{r}{n}}\Big|^{n} = 1 + r + \frac{r^{2}}{2} + \frac{r^{3}}{2.3} + \frac{r^{4}}{2.3.4}$ &c. the amount at the year's end. But this ſeries is the number belonging to the hyperbolic logarithm r, whence

10,	11	The number belonging to the lo-logarithm .43429448r = amount of 1 *l.* for a year = 1.0513; and for 100 = 105.13, to gain intereſt continually.

Schol. If the intereſt for a day be required, ſo that it may amount to $1 + r$ at the year's end, at compound intereſt; then the amount at 1 day's end, will be $\sqrt[n]{1+r}$; which is ſomething leſs than $1 + \frac{r}{n}$.

PROB. XXXIX.

A man puts out a ſum of money at 6 per cent. *to continue* 40 *years; and then both principal and intereſt is to ſink. What is that* per cent. *to continue for ever?*

The queſtion amounts to this; if 100 *l.* be paid for an annuity of 6*l.* a year for 40 years, what is that *per cent?*

Put	1	$a = 6$, $p = 100$, $t = 40$, $r =$ rate o 1 *l.* $R = 1 + r$.
Prob. 34. cor. 2.	2	$t = \frac{\log:a - \log:\overline{a - pr}}{\log:R}$.
2 ×	3	$\text{Log:}\,R = \frac{\log:a - \log:\overline{a-p}}{t}$ $= \frac{l:6 - l:\overline{6 - 100r}}{40}$.
Suppoſe	4	$R = 1.05$; then $r = .05$, and $L:R = .019454$; whence $R =$ 1.046 which is too little.
Suppoſe	5	$R = 1.053$, then $r = .053$, and $L:R = .023324$, and $R = 1.055$ too big.

Then

hen by Rule 5, Prob. xcii, B. I. you will find = 1.052, and the rate = 5.2 *per cent.* which may e repeated for more exactneſs.

PROB. XL.

200 l. *be due* 3 *years hence*; *and* 80 l. 5 *years hence*; *in what time muſt both be paid together, at* 5 per cent.

Let	1	t = the time.
rob. 32. cor. 1.	2	$\frac{200}{\overline{1.05}\|^{5}} = 172.76$, the preſent worth of 200 *l.*
ib.	3	$\frac{80}{\overline{1.05}\|^{5}} = 62.68$, the preſent worth of 80 *l.*
2 + 3	4	235.44, the whole preſent worth.
ob. 32. or. 2.	5	$t = \frac{\log: 280 - \log: 235.44}{\log: 1.05} = 3.5527$ years.

PROB. XLI.

hat muſt I pay for an annuity of 70 l. *to begin* 6 *years hence, and then to continue for* 21 *years, at* 5 per cent?

Let	1	$a = 70$, $t = 21$, $R = 1.05$, $x = 6$.
b. 34.	2	$\frac{1 - \frac{1}{R}}{r} a$ = preſent worth of the annuity 7 years hence = s.
b. 32. r. 1.	3	$\frac{s}{R^{x}} = \frac{1 - \frac{1}{R^{t}}}{rR^{x}} a = \frac{R^{t} - 1}{rR^{t+x}} a$ = preſent worth of s, 7 yeas hence, = 669.704 *l.* the preſent worth of the annuity in reverſion.

SECT. III.

Arithmetical and geometrical Progreſſion.

PROB. XLII.

A traveller ſets out and goes 9 miles a day; 3 days after, another follows him, who travels the firſt day 2 miles, the ſecond 3, the third 4, and ſo on. In what time will he overtake the firſt?

	1	$x =$ days the laſt travelled.
per queſt.	2	$\overline{x - 1 + 2} =$ his laſt day's travel.
arith. progreſſion.	3	$\frac{\overline{x - 1 + 4}}{2} \times x =$ his whole journey.
per qu. {	4	$x + 3 =$ days the firſt travelled.
	5	$\overline{x + 3} \times 9 =$ firſt man's journey.
3 = 5	6	$\frac{xx + 3x}{2} = 9x + 27$
6 reduced	7	$xx - 15x = 54.$
7 extr.	8	$x = 18.$

PROB. XLIII.

There are three numbers in arithmetic progreſſion, th ſquare of the firſt together with the product of th other two is 16; and the ſquare of the mean together with the product of the extreams is 17 What are the numbers?

Put	1	$a - e,\ a,\ a + e$ for the numbers $b = 16,\ c = 17.$
per qu. {	2	$2aa - ae + ee = b$
	3	$2aa \quad - ee = c$

2 +

2 + 3	4	$4aa - ae = b + c = s$ by ſubſt.
4 tran.	5	$ae = 4aa - s$
3 tran.	6	$ee = 2aa - c$
6 × *aa*	7	$aaee = 2a^4 - caa$
5 ⊚ 2	8	$aaee = 16a^4 - 8sa^2 + ss$
7 = 8	9	$16a^4 - 8sa^2 + ss = 2a^4 - caa$
9 reduced	10	$14a^4 - \overline{8s - c}.aa + ss = 0$
10 ext.	11	$aa = 9,\ a = 3.$
5 ÷ *a*	12	$e = \frac{4aa - s}{a} = 1.$
1,	13	and the numbers are 2, 3, 4.

PROB. XLIV.

There are four numbers in arithmetical progreſſion, whoſe common difference is 2, *and product* 3465.

Let	1	$2b = 2$, or $b = 1$, $p = 3465$; $a - 3b$, $a - b$, $a + b$, $a + 3b$, the numbers ſought.
per queſt.	2	$\overline{aa - 9bb} \times \overline{aa - bb} = p$
2 ×	3	$a^4 - 10bba^2 + 9b^4 = p.$
3 extr.	4	$aa = 64,\ a = 8.$
1,	5	the numbers are 5, 7, 9, 11.

PROB. XLV.

To find five numbers in arithmetic progreſſion, whoſe ſum, and product are given.

Put	1	$a - 2e$, $a - e$, a, $a + e$, $a + 2e$ for the numbers, b = ſum = 25, p = product = 2520.
	2	$5a - 3e + 3e = 5a = b.$
per qu.	3	$a = \frac{b}{5} = m$ by ſubſt.
	4	$a \times \overline{aa - 4ee} \times \overline{aa - ee} = p.$

3, 4,	5	$m \times \overline{mm - 4ee} \times \overline{mm - ee} = p.$
5 ×	6	$m \times \overline{m^4 - 5mmee + 4e^4} = p$
6 ÷ m	7	$4e^4 - 5mmee + m^4 = \frac{p}{m}.$
7 extr.	8	$ee = 1,\ e = 1.$
1	9	and the numbers are 3, 4, 5, 6, 7.

PROB. XLVI.

To find three numbers in geometrical progreſſion, where the ſum is 20, *and the ſum of their ſquares* 140.

Let	1	x, y, z be the numbers, $b = 20$, $c = 140$.
per qu:	2	$xz = yy$
	3	$x + y + z = b$
	4	$xx + yy + zz = c$
3 — y	5	$x + z = b - y$
5 @ 2	6	$xx + 2xz + zz = bb - 2by + yy$
6, 2	7	$xx + zz + 2yy = bb - 2by + yy$
7 — yy	8	$xx + zz + yy = bb - 2by$
3, 8	9	$bb - 2by = c.$
9 ÷	10	$y = \frac{bb - c}{2b} = 6\frac{1}{2}.$
2 × 4	11	$4xz = 4yy$
6 — 11	12	$xx - 2xz + zz = bb - 2by - 3yy$
12 ɯ 2	13	$x - z = \sqrt{bb - 2by - 3yy}$
5 + 13	14	$x = \frac{b - y + \sqrt{bb - 2by - 3yy}}{2} = \frac{13\frac{1}{2} + \sqrt{13\frac{1}{4}}}{2}.$
5 — 13	15	$z = \frac{b - y - \sqrt{bb - 2by - 3yy}}{2} = \frac{13\frac{1}{2} - \sqrt{13\frac{1}{4}}}{2}.$

PROB.

PROB. XLVII.

To find four numbers in geometrical progreſſion, whoſe ſum is 15, *and the ſum of their ſquares* 85.

Let	1	v, x, y, z be the numbers, $b=15$, $c=85$.
per qu. {	2	$v+x+y+z=b.$
	3	$v^2+x^2+y^2+z^2=c.$
2 ⊚ 2	4	$\overline{v+x+y+z}\vert^2=\overline{x+y}\vert^2+\overline{v+z}\vert^2+2\times\overline{x+y}\times\overline{v+z}=xx+2xy+yy+vv+2vz+zz+2\times\overline{x+y}\times\overline{v+z}=bb.$
3, 4	5	$c+2xy+2vz+2\times\overline{x+y}\times\overline{v+z}=bb$
by propor.	6	$vz=xy.$
Put	7	$a=x+y$, $e=xy=vz$ by proportion.
2, 7	8	$y+z=b-a.$
5, 7, 8	9	$c+4e+2a\times\overline{b-a}=bb.$
But	10	$v=\frac{xx}{y}$, $z=\frac{yy}{x}$, by the nature of proportion.
2, 10	11	$\frac{xx}{y}+x+y+\frac{yy}{x}=b$
7, 11	12	$a+\frac{x^3+y^3}{xy}=b$
7 ⊚ 3	13	$\overline{x+y}\vert^3=a^3=x^3+3x^2y+3xy^2+y^3$
13 tran.	14	$x^3+y^3=a^3-3xy\times\overline{x+y}$
14 ÷, 7	15	$\frac{x^3+y^3}{xy}=\frac{a^3}{xy}-3a=\frac{a^3}{e}-3a.$
12, 15	16	$a+\frac{a^3}{e}-3a=b.$
16 × e	17	$a^3-2ae=be$
17 tran.	18	$be+2ae=a^3$

18 ÷

18 ÷	19	$e = \frac{a^3}{b + 2a}$
9, 19	20	$c + \frac{4a^3}{b + 2a} + 2ab - 2aa = bb$
20 reduc.	21	$baa + ca = \frac{bb - c}{2} b$
21 extr.	22	$a = 6,$
19	23	$e = 8.$
7	24	$y = \frac{e}{x}, \ a = x + \frac{e}{x}$
24 ×	25	$ax = xx + e$
25 reduc	26	$xx - ax + e = 0.$
26 extr.	27	$x = 2.$
10, 24	28	$y = 4, \ v = 1, \ z = 8.$

PROB. XLVIII.

To find four numbers in geometrical progreſſion, ſuch that the difference of the means is 100, *and the difference of the extreams* 620.

Let	1	a, e, u, y be the numbers, $b = 100$, $c = 620$.
per queſt.	2	$y = a - c, \ u = e - b.$
But	3	$au = ee, \ ay = eu$, by the nature of progreſſion.
3, 2	4	$ae - ab = ee, \ aa - ac = ee - eb.$
4 ÷	5	$a = \frac{ee}{e - b}$
4, 5	6	$aa - ac = \frac{e^4}{\overline{e - b}\vert^2} - \frac{cee}{e - b} = ee - eb.$
6 ×	7	$e^3 - ce \times \overline{e - b} = \overline{e - b}\vert^3$
7 tran.	8	$\overline{3b - c}.ee + \overline{cb - 3bb}.e = -b^3.$
8 red.	9	$ee - be = \frac{b^3}{c - 3b}.$
9 extr.	10	$e = 125$
2, 5	11	$a = 625, \ y = 5, \ u = 25.$

PROB. XLIX.

The ſum of four quantities in geometrical progreſſion being given, and the ſum of the ſquares of the means, to find the quantities.

Let | 1 | a^3, a^2e, ae^2, e^3 be the quantities, $b =$ ſum of all, $c =$ ſum of the ſquares of the means.

| 2 | $a^3 + a^2e + ae^2 + e^3 = b$.

| 3 | $a^4e^2 + a^2e^4 = c$.

| 4 | $a^3 + a^2e + ae^2 + e^3 = \overline{a^2+e^2} \times \overline{a+e} = \dfrac{\overline{a^4e^2 + a^2e^4} \times \overline{a^2e + ae^2}}{a^3e^3} = b$.

Put | 5 | $y = a^2e + ae^2$,

5 & 2 | 6 | $yy = a^4e^2 + 2a^3e^3 + a^2e^4$

6, | 7 | $a^3e^3 = \dfrac{yy - c}{2}$.

3, 4, 5 | 8 | $\dfrac{cy}{\frac{1}{2}yy - \frac{1}{2}c} = b$

8 red. | 9 | $byy - 2cy = bc$

7, | 10 | $ae = \sqrt[3]{\dfrac{yy-c}{2}} = d$

5, | 11 | $a + e = \dfrac{y}{ae} = \dfrac{y}{d}$

10 ÷ a | 12 | $e = \dfrac{d}{a}$

11, 12 | 13 | $a + \dfrac{d}{a} = \dfrac{y}{d}$.

| 14 | $aa - \dfrac{ya}{d} + d = 0$, whence a, e, and all the numbers are known.

PROB. L.

There is given the ſum of the ſquares of the extreams (b), of four quantities in geometrical progreſſion; and the ſum of the means, or their ſquares (c); to find the quantities.

Let		$a^3,\ a^2e,\ ae^2,\ e^3$ be the quantities;
per qu.	2	$a^6 + e^6 = b.$
	3	$a^4e^2 + a^2e^4 = c.$
2 + 3	4	$a^6 + a^4e^2 + a^2e^4 + e^6 = b + c = d.$
	5	$\overline{a^2 + e^2} \times \overline{a^2 + e^2} = \overline{aa + ee}\vert^2$
5 × a^4e^4	6	$\overline{a^6 + a^4e^2} \times \overline{a^2e^4 + e^6} = \overline{a^4e^2 + a^2e^4}\vert^2$
Put	7	$y = a^6 + a^4ee$
4 — 7	8	$a^2e^4 + e^6 = d - y$
3, 6, 7, 8	9	$y \times \overline{d - y} = cc,$
3, 7	10	$a^6 + 2a^4ee + aae^4 = y + c$
10 lw 2	11	$a^3 + aee = \sqrt{y + c} = p.$
3, 4, 7,	12	$a^4ee + 2a^2e^4 + e^6 = c + d - y$
12 lw 2	13	$a^2e + e^3 = \sqrt{c + \overline{d - y}} = q$
11 + 13	14	$a^3 + a^2e + ae^2 + e^3 = p + q.$ Whence the numbers will be found as in the laſt problem.

Or thus,

Let	1	$\frac{xx}{y},\ x,\ y,\ \frac{yy}{x}$ be the quantities.
per qu.	2	$\frac{x^4}{yy} + \frac{y^4}{xx} = b$
	3	$x + y = c,$
3 — x	4	$y = c - x$
2 × $yyxx$	5	$x^6 + y^6 = bxxyy$
4, 5	6	$x^6 + \overline{c - x}^6 = bxx \times \overline{c - x}\vert^2$

6	7	$x^6 + c^6 - 6c^5x + 15c^4xx - 20c^3x^3 + 15c^2x^4 - 6cx^5 + x^6 = bccx^2 - 2bcx^3 + bx^4$
reduced	8	$\begin{array}{llllll} 2x^6 & - 6cx^5 & + 15ccx^4 & - 20c^3x^3 & & \\ & & - b & + 2bc & & \\ & + 15c^4xx & - 6c^5x & & + c^6 = 0. & \\ & - bcc & & & & \end{array}$

PROB. LI.

Given the ſum of the extreams (b) of five quantities in geometrical progreſſion, and the ſum of the three means (c), to find the quantities.

Let	1	a^4, a^3e, a^2e^2, ae^3, e^4 be the quantities.
per qu.	2	$a^4 + e^4 = b.$
	3	$a^3e + a^2e^2 + ae^3 = c.$
	4	$a^2e^2 \times \overline{a^4 + 2a^2e^2 + e^4} = a^2e^2 \times \overline{aa + ee}\|^2 = \overline{a^3e + ae^3}\|^2.$
Put	5	$y = aaee.$
2, 4, 5	6	$y \times \overline{b + 2y} = \overline{c - y}\|^2$
6 reduced	7	$yy + by + 2cy = cc.$
2 ± 5	8	$a^4 \pm 2aaee + e^4 = b \pm 2y$
8 lω 2	9	$aa + ee = \sqrt{b + 2y}$
	10	$aa - ee = \sqrt{b - 2y}$ Whence a, e, and all the quantities are eaſily found.

PROB.

PROB. LII.

Of five quantities in geometrical progreſſion, there is given the ſum of the extreams (b), and the ſum of the ſquares of the three means (c); to find the proportionals.

Let	1	a^4, a^3e, a^2e^2, ae^3, e^4, be the quantities.
per qu.	2	$a^4 + e^4 = b.$
	3	$a^6e^2 + a^4e^4 + a^2e^6 = c.$
	4	$a^2e^2 \times \overline{a^4 + e^4} = a^6ee + a^2e^6$
Put	5	$y = aaee$
2, 3, 4, 5	6	$y \times b = c - a^4e^4 = c - yy.$
$6 + yy$	7	$yy + by = c.$
2, 5	8	$a^4 \pm 2a^2ee + e^4 = b \pm 2y$
8 lω 2	9	$aa + ee = \sqrt{b + 2y}.$
	10	$aa - ee = \sqrt{b - 2y}.$
		whence all the reſt are found.

PROB. LIII.

There are four quantities in geometrical proportion diſcreet, whoſe ſum is b, ſum of the ſquares c, and ſum of their cubes d; to find the numbers.

Let	1	x, ex, y, ey, be the numbers.
per qu.	2	$x + ex + y + ey = b$
	3	$x^2 + e^2x^2 + y^2 + e^2y^2 = c$
	4	$x^3 + e^3x^3 + y^3 + e^3y^3 = d$
Put	5	$x + y = v$, $x^2 + y^2 = z.$
	6	$1 + e = s$, $1 + ee = t$, $1 + e^3 = u$
5 ⊚ 2&c.	7	$xy = \frac{vv - z}{2}$, $x^3 + y^3 = \frac{2z - vv}{2}\, v$
2, 5, 6,	8	$sv = b$

3, 5, 6

3, 5, 6	9	$tz = c$
4, 6, 7	10	$\frac{3z - vv}{2} vu = d.$
8, 9	11	$v = \frac{b}{s}, \; z = \frac{c}{t}$
10, 11	12	$\frac{3bcu}{2st} - \frac{b^3u}{2s^3} = d$
reduced	13	$3bcs^2u - b^3tu = 2s^3td.$
6, 13,	14	and reſtoring the values of s, t, u; then $3bc \times \overline{1 + e^3} \times \overline{1 + e}\vert^2 - b^3 \times \overline{1 + e^3} \times \overline{1 + ee} = 2d \times \overline{1 + e}\vert^3 \times \overline{1 + ee}$, a 5th power. And e being known all the reſt are eaſily found.

◆

SECT IV.

Unlimited Problems.

PROB. LIV.

How many old guineas at 21 s. 6 d. *and piſtoles at* 17 s. *will pay* 100 l; *and how many ways can it be done?*

Let	1	a = guineas, e = piſtoles; 21 *s.* 6 *d.* = 43 ſix-pences, 17 *s.* = 34 ſix-pences, and 100 *l.* = 4000 ſix-pences.
per queſt.	2	$43a + 34e = 4000$
$2 -$	3	$34e = 4000 - 43a.$
$3 \div$	4	$e \sqsupset 116^{1}\frac{3}{4}.$
$2 \div$	5	$a = \frac{4000 - 34e}{43} = wh.$
5 abridg.	6	$\frac{1 - 34e}{43} = wh.$
$\frac{43e}{43} + 6$	7	$\frac{9e + 1}{43} = wh.$
$7 \times (5)$	8	$\frac{45e + 5}{43} = wh.$
$8 - \frac{43e}{43}$	9	$\frac{2e + 5}{43} = wh.$
$9 \times (4)$	10	$\frac{8e + 20}{43} = wh.$
$7 - 10$	11	$\frac{e - 19}{43} = wh. = p.$
$11 \times (43)$	12	$e = 43p + 19 = 19, 62, 105,$ the piſtoles.
5	13	$a =$ 78, 44, 10, the guineas; being three anſwers in whole numbers.

PROB.

PROB. LV.

What number is that, which being divided by 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, *there will remain* 1; *but divided by* 13, *then* 0 *will remain.*

	1	It is plain $5 \times 7 \times 8 \times 9 \times 11$, or 27720 is diviſible by any of theſe numbers.
Put {	2	$a =$ ſome whole number.
	3	$13e =$ the number ſought.
per queſt.	4	$27720a + 1 = 13e$
4 ÷	5	$e = \frac{27720a + 1}{13} = wh.$
5 abrid.	6	$\frac{4a + 1}{13} = wh.$
6 × (3)	7	$\frac{12a + 3}{13} = wh.$
(1) — 7	8	$\frac{a - 3}{13} = wh. = p.$
8 ×	9	$a = 13p + 3 = 3$, 16, &c.
5,	10	$e = 6397$.
10 × (13)	11	$13e = 83161$ the number ſought.

PROB. LVI.

A man bought 20 *birds for* 20 *pence; geeſe at* 4 d. *quails at half pennies, and larks at farthings. How many did he get of each?*

Let	1	$a =$ geeſe, $e =$ quails, $y =$ larks.
per qu. {	2	$a + e + y = 20$.
	3	$4a + \frac{1}{2}e + \frac{1}{4}y = 20$.
2 tr.	4	$y = 20 - a - e$
3, 4	5	$4a + \frac{1}{2}e + 5 - \frac{a + e}{4} = 20$.
5 red.	6	$15a + e = 60$

6,	7	$a \sqsupset 4.$
2 tran.	8	$e = 20 - a - y$
3, 8	9	$4a + 10 + \frac{1}{4}y - \frac{a+y}{2} = 20$
9 red.	10	$7a - \frac{1}{2}y = 20.$
10	11	$a \sqsubset 2.$
7, 11	12	$a = 3$
2, 12	13	$e + y = 17.$
3, 12	14	$\frac{1}{2}e + \frac{1}{4}y = 8.$
14 × (2)	15	$e + \frac{1}{2}y = 16$
13 — 15	16	$\frac{1}{2}y = 1,\ y = 2.$
13,	17	$e = 15.$

PROB. LVII.

A, B, C, *and their wives* P, Q, R, *went to th market to buy hogs. Each man and woman bough as many hogs as they gave ſhillings for each hog* A *bought* 23 *hogs more than* Q; *and* B *bough* 11 *more than* P. *Alſo each man laid out* 3 *guineas more than his wife. Which* 2 *perſons wer man and wife.*

Let	1	x = hogs ſome man bought, $x - y$ = wife's hogs.
	2	the money for the man's $= xx$,
per qu.	3	$xx - 2xy + yy$ = wife's money.
	4	$xx = xx - 2xy + yy + 63.$
4 tr.	5	$2xy = 63 + yy$
5 ÷	6	$x = \frac{63 + yy}{2y} = wh.$
6 — y	7	$x - y = \frac{63 - yy}{2y} = wh.$
But	8	In this caſe y muſt be an od number, = 1, 3, 5, 7, &c. bu it cannot be 5.

8,	9	If $y=1$, $x-y=31$, $x=32$ $y=3$, $x-y=9$, $x=12$, $y=7$, $x-y=1$, $x=8$
per queſt.	10	A has 32 hogs, and Q 9, Alſo B has 12, and P 1,
	11	Whence B and Q C and P A and R } are man and wife.

PROB. LVIII.

To find e, y in whole numbers, ſo that $yy-ee+22e=184$.

	1	To depreſs the equation, put $y=x+e$.
	2	$yy-ee+22e=xx+2xe+22e=184$.
2 ÷	3	$e=\frac{184-xx}{22+2x}=\frac{92-\frac{1}{2}xx}{11+x}=wh.$ therefore x muſt be an even number. Therefore
2 ÷	4	rather make $2e=\frac{184-xx}{11+x}=wh.$
4,	5	reduce the equation as low as it can, then $2e=-x+11+\frac{63}{11+x}=wh.$
5,	6	$\frac{63}{11+x}=wh.$
Put	7	$11+x=p$
6, 7	8	$\frac{63}{p}=wh.$ Therefore take $p=$ any diviſor of 63, that is $p=1, 3, 7, 9, 21, 63$.
7, 8	9	then $x=-10, -8, -4, -2, 10, 52$
8,	10	$\frac{63}{p}=63, 21, 9, 7, 3, 1$
5, 9	11	$2e=84, 40, 24, 20, 4, -40$.
11 ÷ (2)	12	$e=42, 20, 12, 10, 2, -20$.
1,	13	$y=32, 12, 8, 8, 12, 32$.

And any pair of theſe will ſolve the problem, whic are all the poſſible anſwers in whole numbers.

PROB. LIX.

A vintner has wine at 24 d. 22 d. *and* 18 d. pe *gallon; of which he would mix* 30 *gallons, to b ſold at* 20 d. *How much muſt he take of each?*

Let	1	a, e, y be the quantities of each.
per qu.	2	$a + e + y = 30.$
	3	$24a + 22e + 18y = 600.$
$2 \times (24)$	4	$24a + 24e + 24y = 720.$
$2 \times (22)$	5	$22a + 22e + 22y = 660.$
$4 - 3.$	6	$2e + 6y = 120.$
$6 \div (2)$	7	$e + 3y = 60 \,.\, e = 60 - 3y.$
7,	8	$y \sqsupset 20.$
$5 - 3$	9	$-2a + 4y = 60 \,.\, a = 2y - 30.$
$9 + 2a$	10	$4y = 60 + 2a.$
10,	11	$y \sqsubset 15.$
8, 10,	12	$y = 16,\ 17,\ 18,\ 19.$
9,	13	$a = \ 2,\ \ 4,\ \ 6,\ \ 8.$
7,	14	$e = 12,\ \ 9,\ \ 6,\ \ 3.$

PROB. LX.

To find the value of e, y, u, x, z in whole numbe in the two given equations following.

Given	1	$e + y + u + x + z = 60.$
	2	$3e + 4y + 5u + 7x + 9z = 440.$
$1 \times (4)$	3	$4e + 4y + 4u + 4x + 4z = 240$
$1 \times (9)$	4	$9e + 9y + 9u + 9x + 9z = 540$
$2 - 3$	5	$-e \quad + u + 3x + 5z = 200. e \sqsubset$
$4 - 2$	6	$6e + 5v + 4u + 2x \ = 100. e \sqsupset 1$
Suppoſe	7	$e = 10.$
$i - e$	8	$y + u + x + z = 60 - e = 50$

2 —

$2 - 3e$	9	$4y+5u+7x+9z=440-3e=410.$
$8 \times (5)$	10	$5y+5u+5x+5z=250.$
$8 \times (9)$	11	$9y+9u+9x+9z=450.$
$9 - 10$	12	$-y \quad +2x+4z=160 . y \sqsubset 0.$
$11 - 9$	13	$5y+4u+2x \quad = 40 . y \sqsupset 6\frac{4}{5}.$
Suppoſe	14	$y = 4.$
$8 - y.$	15	$u + x + z = 50 - y = 46.$
$9 - 4y$	16	$5u + 7x + 9z = 410 - 4y = 394.$
$15 \times (5)$	17	$5u + 5x + 5z = 230.$
$15 \times (7)$	18	$7u + 7x + 7z = 322$
$16 - 17$	19	$2x + 4z = 164 . z \sqsupset 40\frac{1}{2}.$
$16 - 18$	20	$-2u \quad + 2z = 72 . z =$ or $\sqsubset 37.$
Suppoſe	21	$z = 40.$
$15 - z$	22	$u + x = 46 - z = 6.$
$16 - 9z$	23	$5u + 7x = 394 - 9z = 34$
$22 \times (5)$	24	$5u + 5x = 30$
$23 - 24$	25	$2x = 4 . x = 2$
$22 - x$	26	$u = 6 - x = 4.$

nd one anſwer is got, *viz.* $e = 10, y = 4, u = 4,$ $= 2, z = 40.$ for $10 + 4 + 4 + 2 + 40 = 60.$
nd $3 \times 10 + 4 \times 4 + 5 \times 4 + 7 \times 2 + 9 \times 40 = 440.$

PROB. LXI.

o find a perfect number, or one which is equal to the ſum of all its aliquot parts.

Suppoſe	1	$y^n x =$ a perfect number.
	2	then $1 + y + y^2 \ldots$ to y^n, $+ x + xy + xy^2 \ldots$ to $xy^{n-1} =$ ſum of all the aliquot parts.
r queſt.	3	$y^n x = 1 + y \ldots y^n + x \times \overline{1 + y \ldots y^{n-1}}$

Cor. 3. Pr. 26 propor.	4	$1 + y \ldots y^n = \frac{y^{n+1} - 1}{y - 1}$
	5	$1 + y \ldots y^{n-1} = \frac{y^n - 1}{y - 1}$.
3, 4, 5	6	$y^n x = \frac{y^{n+1} - 1}{y - 1} + \frac{y^n - 1}{y - 1} x$
6 × tr.	7	$\overline{y - 1}\,.\,y\ x - \overline{y^n - 1}\,.\,x = y^{n+1} - 1$
7 ÷	8	$x = \frac{y^{n+1} - 1}{\overline{y - 1}\,.\,y^n - y^n + 1}$.
	9	But that x may be a whole number, $\overline{y - 1}\,.\,y^n - y^n + 1 = 1$.
9 tr.	10	$\overline{y - 1}\,.\,y^n = y^n$
10 ÷	11	$y - 1 = 1$
11 tr.	12	$y = 2$.
1, 12	13	$2^n x =$ a perfect number.
8, 13	14	$2^n \times \overline{2^{n+1} - 1} =$ perfect number, where $2^{n+1} - 1$ muſt be a prime as appears by ſtep 2.
	15	If n is an odd number greater than 1, then $2^{n+1} - 1$ will be a compoſite number.

SECT

SECT. V.

Rational Squares, Cubes, &c.

PROB. LXII.

To find two ſquare numbers, whoſe difference is given.

	1	Let xx and yy be the numbers, $a =$ difference.
Put	2	$\frac{z+v}{2} = x$, $\frac{z-v}{2} = y$
2 © 2	3	$\frac{zz+2zv+vv}{4} = xx$
2 © 2	4	$\frac{zz-2zv+vv}{4} = yy$
3 — 4	5	$zv = xx - yy$
1, 5	6	$zv = a$
	7	Take v at pleaſure, then $z = \frac{a}{v}$, whence x and y are known.
	8	If a is a whole number, and x and y are deſired in whole numbers; take any two factors that produce a, ſo they be both even or both odd numbers, if poſſible. And therefore a muſt be either an odd number greater than 1, or a number diviſible by 4, to have x and y in whole numbers.

If $a = 27$. Take $v=1$, $z=27$, or $v=3$, $z=9$.
If $a = 20$. $v=2$, $z=10$.

PROB. LXIII.

To divide a given ſquare into two other ſquares.

Let	1	aa, ee = the ſquares required, bb = the given ſquare.
aſſume	2	$a = sv$
	3	$e = rv - b$
2 @ 2	4	$aa = ssvv$
3 @ 2	5	$ee = rrvv - 2rbv + bb$
4 + 5	6	$aa + ee = \overline{rr + ss}.vv - 2rbv + bb$
per queſt.	7	$\overline{rr + ss}.vv - 2rbv + bb = bb$
7 tr.	8	$\overline{rr + ss}.vv = 2rbv$
8 ÷	9	$v = \frac{2rb}{rr + ss}$.
2, 9	10	$a = \frac{2rbs}{rr + ss}$.
3, 9	11	$e = \frac{rr - ss}{rr + ss} b$

PROB. LXIV.

To find a ſquare number (aa), which multiplied by a given number (n), and a given ſquare (bb) added to it; the ſum may be a ſquare.

Let	1	$naa + bb = yy$.
aſſume	2	$va \backsim b = y$
2 @ 2	3	$v^2aa - 2bva + bb = yy$
$1 = 3$	4	$v^2aa - 2bva + bb = naa + bb$
4 tr.	5	$\overline{vvaa - naa} = 2bva$
$5 \div a$	6	$\overline{vv - n}.a = 2bv$
	7	$a = \frac{2bv}{vv - n}$, where v may be taken at pleaſure.

PROB.

PROB. LXV.

To find two ſquare numbers (aa, ee), that their product added to a given number (d), may be a ſquare.

Let aſſume	1	$aaee + d = yy$
	2	$ae - v = y$
2 ◎ 2	3	$aaee - 2aev + vv = yy$
1 = 3	4	$aaee + d = aaee - 2aev + vv$
4 tr.	5	$2aev = vv - d$
	6	$c = \frac{vv - d}{2av}$, where a and v may be taken at pleaſure.

PROB. LXVI.

To find three ſuch numbers x, y, z; ſo that yy = xz; and x + y, and z + y, may be two ſquares.

Aſſume {	1	$x + y = aa$
	2	$z + y = ee$
1 − y	3	$x = aa - y$
2 − y	4	$z = ee - y$
3 × 4	5	$xz = \overline{aa - y} \times \overline{ee - y} = yy$ *per* queſt.
5 ×	6	$yy = a^2e^2 - aay - eey + yy.$
6 tr.	7	$a^2e^2 = \overline{aa + ee}.y$
	8	$y = \frac{aaee}{aa + ee}$, where a, e, may be taken at pleaſure.
3, 8	9	$x = aa - \frac{aaee}{aa + ee} = \frac{a^4}{aa + ee}$
4, 8	10	$z = ee - \frac{aaee}{aa + ee} = \frac{e^4}{aa + ee}.$

PROB. LXVII.

To find a number, from which two given numbers (a, b) being ſeverally ſubtracted; the remainders ſhall be two ſquares.

Let	1	$x =$ the number ſought.
per qu.	2	$x - a = yy$
	3	$x - b = zz$
$2 + a$	4	$x = a + yy$
3, 4	5	$a + yy - b = zz$
aſſume	6	$v - y = z$
6 ⊚ 2	7	$vv - 2vy + yy = zz$
5 = 7	8	$a - b + yy = vv - 2vy + yy$
8 tr.	9	$2vy = vv + b - a$
$9 \div 2v$	10	$y = \frac{vv + b - a}{2v}.$
4, 10	11	$x = a + \frac{\overline{vv + b - a}\vert^{2}}{4vv}$

PROB. LXVIII.

To find three numbers (x, y, z), whoſe ſum ſhall be a ſquare, and alſo the ſum of any two to be a ſquare.

Suppoſe	1	$x + y = rr$
	2	$x + z = ss$
	3	$y + z = tt$
	4	$x + y + z = vv$
$4 \times (2)$	5	$2vv = 2x + 2y + 2z$
$1 + 2 + 3$	6	$2x + 2y + 2z = rr + ss + tt$
5 = 6	7	$2vv = rr + ss + tt$
Put	8	$s = p - v,\ t = q - v$
7, 8	9	$2vv = rr + pp - 2pv + vv$
		$+ qq - 2qv + vv$
9 tr.	10	$2pv + 2qv = rr + pp + qq$

	11	$v = \frac{rr + pp + qq}{2p + 2q}$, where r, p, q are taken at pleaſure; whence s and t are known (ſtep 8).
4—3	12	$x = vv - tt$
4—2	13	$y = vv - ss$
4—1	14	$z = vv - rr$

PROB. LXIX.

To find three ſquares in arithmetic proportion.

Suppoſe	1	$xx - y = vv$
	2	xx
	3	$xx + y = zz$
1 + 3	4	$2xx = vv + zz$
aſſume	5	$v = s - x$
	6	$z = t - x$
4, 5, 6	7	$2xx = ss - 2sx + xx + tt - 2tx + xx$
7 tr.	8	$\overline{2s + 2t}.x = ss + tt$
8 ÷	9	$x = \frac{ss + tt}{2s + 2t}$.
5, 9	10	$v = s - x = \frac{ss + 2st - tt}{2s + 2t}$.
6, 9	11	$z = t - x = \frac{2st + tt - ss}{2s + 2t}$.

PROB. LXX.

To find two numbers (x, y) ſo that $xy + x$, and $xy + y$, may be ſquares.

per qu.	1	$xy + x = vv$
	2	$xy + y =$ a ſquare.
1 ÷ x	3	$y + 1 = \frac{vv}{x}$.

3 — 1

$3 - 1$	4	$y = \frac{vv - x}{x}$
2, 4	5	$\frac{vv - x}{x} \times \overline{x + 1} = \square$
5,	6	to effect this, let $x + 1$ be the ſide, then $\frac{vv - x}{x} = x + 1$
6 red.	7	$vv = xx + 2x =$ ſquare.
Let	8	$r - x =$ ſide,
7, 8	9	$rr - 2rx + xx = xx + 2x$
9 red.	10	$x = \frac{rr}{2r + 2}$.
4, 6, 10	11	$y = 1 + \frac{rr}{2r + 2}$, where r may be taken at pleaſure.

PROB. LXXI.

To find two numbers, whoſe ſum and difference, ſhall be two ſquares.

Let	1	a, e, be the numbers.
per qu. {	2	$a + e = yy$
	3	$a - e =$ a ſquare.
$2 - e$	4	$a = yy - e$
3, 4	5	$yy - 2e =$ a ſquare.
Put	6	$r - y =$ root of it.
5, 6	7	$rr - 2ry + yy = yy - 2e$
7 tr.	8	$2ry = rr + 2e$
$8 \div 2r$	9	$y = \frac{rr + 2e}{2r}$, where r, e may be taken at pleaſure; then
4,	10	$a = yy - e$.

PROB.

PROB. LXXII.

To find three numbers (a, e, y), that the ſum of their ſquares may be a ſquare.

per queſt.	1	$aa + ee + yy = vv$
aſſume	2	$v = d + y$
2 @ 2	3	$v^2 = dd + 2dy + yy$
1 = 3	4	$aa + ee + yy = dd + 2dy + yy$
4 ÷	5	$y = \frac{aa + ee - dd}{2d}$, where a, e, and d are taken at pleaſure.

PROB. LXXIII.

To divide a number into two parts, ſo that the ſum of the ſquares may be a ſquare.

Let	1	$s =$ the number; a, e, the parts.
per qu. {	2	$s = a + e$
	3	$aa + ee = vv.$
2 @ 2	4	$aa + 2ae + ee = ss.$
4 —	5	$aa + ee = ss - 2ae$
2 — a	6	$e = s - a$
3, 6, 5	7	$\underline{aa + ee} = ss - 2as + 2aa = vv$
aſſume	8	$ra - s = v.$
7, 8	9	$ss - 2as + 2aa = rraa - 2rsa + ss$
9 reduc.	10	$a = \frac{r - 1}{rr - 2} \times 2s$
6, 10	11	$e = \frac{r - 2}{rr - 2} \times rs.$

PROB.

PROB. LXXIV.

To find two numbers in the ratio of b to c, ſo that either of them added to the ſquare of the other, ſhall make two ſquares.

Let	1	ba, ca, be the numbers.
per qu.	2	$bbaa + ca =$ a ſquare.
	3	$ccaa + ba =$ a ſquare.
Put	4	$bbaa + ca = \overline{ba - v}^2 = bbaa - 2bva + vv$
4 tr.	5	$2bva + ca = vv$.
5 ÷	6	$a = \frac{vv}{2bv + c}$
3, 6	7	$ccaa + ba = \overline{\frac{ccvv}{2bv + c}} + b \times \frac{vv}{2bv + c}$
7,	8	$ccaa + ba = \overline{ccvv + 2bbv + bc} \times \left.\overline{\frac{vv}{2bv + c}}\right\|^2 =$ a ſquare.
8 ÷ □	9	$ccvv + 2bbv + bc =$ a ſquare $= zz$.
Put	10	$z = cv - r$.
9, 10	11	$ccvv + 2bbv + bc = ccvv - 2cvr + rr$
11 reduc.	12	$v = \frac{rr - bc}{2bb + 2cr}$.

PROB. LXXV.

To find a number, to which adding a given cube number, the ſum ſhall be a cube; and ſubtracting another cube number, the remainder ſhall be a cube.

Let	1	x be the number; b^3, c^3 the two cubes.
per qu.	2	$x + b^3 =$ a cube
	3	$x - c^3 =$ a cube

aſſume 4 $x + b^3 = \overline{b + \frac{cc}{bb} a}|^3 = b^3 + 3c^2a + \frac{3c^4}{b^3} aa + \frac{c^6}{b^6} a^3.$

4, 5 $x = 3c^2a + \frac{3c^4}{b^3} aa + \frac{c^6}{b^6} a^3.$

aſſume 6 $x - c^3 = \overline{a - c}|^3 = a^3 - 3a^2c + 3acc - c^3$

6, 7 $x = a^3 - 3a^2c + 3acc$

5 = 7 8 $a^3 - 3a^2c + 3acc = 3acc + \frac{3c^4}{b^3} aa + \frac{c^6}{b^6} a^3.$

8 ÷ 9 $a - 3c = \frac{3c^4}{b^3} + \frac{c^6}{b^6} a.$

9 reduced 10 $a = \frac{b^3 + c^3}{b^6 - c^6} \times 3cb^3 = \frac{3cb^3}{b^3 - c^3}.$

PROB. LXXVI.

To divide the ſum of two given cubes into two other cubes.

Let 1 x^3, y^3 be the cubes ſought; b^3, c^3 the given cubes.

aſſume 2 $x = b + v$

aſſume 3 $y = c - \frac{bb}{cc} v$

2 © 3 4 $x^3 = b^3 + 3bbv + 3bv^2 + v^3$

3 © 3 5 $y^3 = c^3 - 3bbv + \frac{3b^4}{c^3} v^2 - \frac{b^6}{c^6} v^3$

4 + 5 6 $x^3 + y^3 = b^3 + c^3 + 3bv^2 + \frac{3b^4}{c^3} v^2 + v^3 - \frac{b^6}{c^6} v^3 = b^3 + c^3$, *per* queſt.

6 tr.	7	$3bv^2 + \frac{3b^4}{c^3}v^2 = \frac{b^6}{c^6}v^3 - v^3.$
$7 \times \frac{c^6}{vv}$	8	$3bc^6 + 3b^4c^3 = b^6v - c^6v.$
8 ÷	9	$v = \frac{b^3 + c^3}{b^6 - c^6} \times 3bc^3 = \frac{3bc^3}{b^3 + c^3}.$

PROB. LXXVII.

To find three ſuch cube numbers, whoſe ſum may be both a ſquare and a cube number.

Let	1	a, e, y be their roots; x^6 the ſum of their cubes.
per queſt.	2	$a^3 + e^3 + y^3 = x^6$
$2 - y^3$	3	$e^3 + y^3 = x^6 - a^3$
aſſume	4	$\frac{x^4}{aa}v - a = e$
	5	$xx - v = y$
	6	$a^3 =$ a cube
4 @ 3	7	$\left.\begin{array}{l} + \frac{x^{12}}{a^6}v^3 - \frac{3x^8}{a^3}v^2 + 3x^4v - a^3 \\ + x^6 - 3x^4v + 3x^2v^2 - v^3 \end{array}\right\\} = x^6$, *per* qu.
5 @ 3	8	
6, 7, 8,	9	$\frac{x^{12}}{a^6}v^3 - \frac{3x^8}{a^3}vv + 3x^2v^2 - v^3 = 0.$
$9 \times \frac{a^6}{vv}$	10	$x^{12}v - a^6v = \overline{x^8 - x^2a^3} \times 3a^3.$
	11	$v = \frac{x^6 - a^3}{x^{12} - a^6} \times 3a^3x^2 = \frac{3a^3xx}{x^6 + a^3}$, where x and a may be taken at pleaſure; then v being known, e and y are known by ſtep 4 and 5.
Suppoſe		$x = 1$, $a = \frac{1}{2}$, then $v = \frac{1}{3}$, whence $e = \frac{5}{6}$, $y = \frac{2}{3}$, and the numbers are $\frac{1}{8} + \frac{125}{216} + \frac{8}{27} = 1$,
or ſuppoſe		$x = 2$, $a = 1$, then $u = \frac{12}{65}$, and $e = \frac{127}{65}$, $y = \frac{248}{65}$; and the numbers $1 + \frac{2048383}{274625} + \frac{15252992}{274625} = 64.$

SECT. VI.

Geometrical Problems.

Fig.

HAVING hitherto in all the foregoing ſections, kept an account of the whole proceſs by regiſtering the ſeveral ſteps at length in the margin; ſo that the reader may ſee at once how each ſtep is derived from the reſt; and by this means become acquainted with the manner of proceeding, in any operation. It may be preſumed, that by this time, he will be able to ſee the connection of the ſeveral parts of the proceſs in any ſolution, without ſuch a formal explanation. Therefore, for brevity's ſake, in what follows, I ſhall not tie myſelf to this method, but generally write down the proceſs after a ſhorter way, without notifying all theſe particulars; and content myſelf with mentioning only ſuch deductions as are leſs obvious.

PROB. LXXVIII.

In the triangle CAD; *there are given* AC, AD; *and the lines* CE, DB, *drawn to the given points* E, B; *to find the point of interſection* F. 23.

Put $AB = r$, $CB = m$, $AE = p$, $AD = d$, $CE = f$, $DB = g$; and the line ſought $CF = a$. Draw EI ∥ to DB.

By the ſimilar triangles CBF, CIE (Geom. II. 12,) $a : f - a :: m : \frac{mf - ma}{a} = BI$, and $d : p :: r : \frac{pr}{d} = AI$. Then $AI + BI = AB$, that is, $\frac{pr}{d}$ +

Fig. 23. $+\frac{mf-ma}{a}=r$, and multiplying by da, $pra+dmf-dma=dar$, and $dra+dma-pra=dmf$, whence $a=\frac{dmf}{dr+dm-pr}$.

PROB. LXXIX.

24. *To divide a triangle* ABC *in a given ratio, by a line drawn through a given point* P.

Through P draw ED parallel to BA, and put $AB=b$, $AC=d$, $BC=f$, $BE=g$, $EP=p$, $BF=x$, and the ratio as m to n, and $m+n=s$.

By ſimilar triangles, $g+x:p::x:\frac{px}{g+x}=BI$; then (Geom. II. 19.) $BI\times BF:BA\times BC::m:m+n$, that is,

$\frac{pxx}{g+x}:bf::m:s$; then $bmf=\frac{spxx}{g+x}$, and $spxx=bmfx+bmfg$: by which equation x is found.

PROB. LXXX.

25. *To divide a triangle into two equal parts, by a line of a given length.*

Let BD be perpendicular to AC, KH the given line, and HL parallel to BD. Put $AC=a$, $BC=b$, $HK=c$, $CD=d$, $CK=x$; then (Geom. II. 19) $AC\times BC=2KC\times HC$, or $ab=2x\times CH$, and $CH=\frac{ab}{2x}$, and by ſimilar triangles (Geom. II. 13.) $b:d::\frac{ab}{2x}:\frac{abd}{2bx}=CL$; and $KL=KC-CL=x-\frac{abd}{2bx}=\frac{2bxx-abd}{2bx}$. But (Geom. II. 21. cor. 5.)

$HK^2 - KL^2 = CH^2 - CL^2$, or Fig.

$$\frac{4bbccxx - 4bbx^4 - a^2b^2d^2 + 4ab^2dx^2}{4bbxx} = \frac{a^2b^4 - a^2b^2d^2}{4bbx}$$ 25.

or $4bbx^4 - 4bbccx^2$
$- 4adbb = - a^2b^4$, whence x will be found.

PROB. LXXXI.

To find the inacceſſible diſtance AB, *by help of the triangle* ACD; CAB *being one right line.* 26.

Through B draw BEF, and draw EG parallel to CD. Put $AC = a$, $AD = b$, $CD = c$, $AE = d$, $CF = f$, and $AB = x$. Then, by ſimilar triangles, AD (b) : CD (c) : : AE (d) : $EG = \frac{cd}{b}$; and AD (b) : CA (a) : : AE (d) : $AG = \frac{ad}{b}$.

Then $GB = \frac{ad + bx}{b}$. And by the ſimilar triangles BGE, BCF; CF (f) : CB $(a + x)$: : EG $\left(\frac{cd}{b}\right)$: GB $\left(\frac{ad + bx}{b}\right)$. Therefore $\frac{adf + bfx}{b} = \frac{cda + cdx}{b}$, $bfx - cdx = cda - fda$, and $x = \frac{c-f}{bf-cd} da$.

PROB. LXXXII.

If the line EFB *be drawn from the angle* E, *perpendicular to the diagonal* AD *of a right-angled parallelogram, and* BF, FD *are given. To find the ſides of the paralellogram.* 27.

Let $AF = x$, $EF = y$, $BF = b$, $DF = c$. The triangles AFB, AFE, and DFE are ſimilar. Therefore

Fig. 27. fore $b : x :: x : \frac{xx}{b} = \text{FE} = y$, and $b : x :: y$ or $\frac{xx}{b} : c$.

Whence $\frac{x^3}{b} = bc$, and $x^3 = bbc$, and $x = \sqrt[3]{bbc}$.

Then $\text{AE} = \sqrt{xx + yy}$, and $\text{ED} = \sqrt{cc + yy}$.

PROB. LXXXIII.

28. *To deſcribe a ſquare in the given triangle* ATE.

Draw TC perpendicular to AE, and let BFGD be the ſquare. Put $\text{AE} = b$, $\text{AC} = c$, $\text{CE} = d$, $\text{TC} = p$, BF or $\text{BD} = x$. $\text{AB} = y$. Then

The triangles ABF, ACT are ſimilar, and $y : x :: c : p$, whence $cx = py$. Alſo the triangles EDG, ECT are ſimilar, and $\text{ED} = b - x - y$, whence $b - x - y : x :: d : p$, and $dx = pb - px - py = pb - px - cx$. Whence $x = \frac{pb}{d + c + p} = \frac{pb}{b + p}$.

PROB. LXXXIV.

29. *Six equal circles of 2 inches diameter are inſcribed in an equilateral triangle, touching one another and the ſides of the triangle. To find the ſide of the triangle.*

Draw AF perpendicular to BC, and from the centers O, S, draw OD, SE perpendicular to AB, and let $\text{DO} = r$, $\text{AB} = x$.

The triangles ABF, ADO, ESB are ſimilar, and (Geom. II. 39. cor.) $\text{AF} = \text{AB}\sqrt{\tfrac{3}{4}}$. Then BF $(\tfrac{1}{2}x) : \text{AF}\ (x\sqrt{\tfrac{3}{4}} :: \text{DO}\ (r) : \text{AD} = \frac{rx\sqrt{\tfrac{3}{4}}}{\tfrac{1}{2}x} = 2r\sqrt{\tfrac{3}{4}}$ $= \text{EB}$, and $\text{DE} = 4r$, whence AB or $x = 4r + 4r\sqrt{\tfrac{3}{4}} = \frac{2 + \sqrt{3}}{1} \times 2r = 4 + 2\sqrt{3}$.

PROB.

PROB. LXXXV.

There are two circles BDA *and* BFC *touching in* B, *and if* DE *be perpendicular to* BA *at the center* E; *then there is given* AC *and* DF; *to find the diameters.* 30.

Let radius $BE=a$, $DF=b$, $CA=d$; then $FE=a-b$, $EC=a-d$, then $FE^2=BE\times EC$ (Geom. IV. 17), that is, $aa-2ab+bb=aa-ad$, and $2ba-da=bb$, and $a=\frac{bb}{2b-d}$, whence $BC=2a-d$.

PROB. LXXXVI.

In the triangle ABC, *there are given the three perpendiculars, from the angles upon the oppoſite ſides; to find the ſides.* 31.

Let $AO=a$, $CP=b$, $BR=c$, and $AB=y$. Then twice the area $=by=AC\times c=CB\times a$, whence $AC=\frac{by}{c}$, and $CB=\frac{by}{a}$. And (Geom. II. 21.) $\frac{bbyy}{cc}-bb=AP^2$; and (Geom. II. 23. cor.)

$$AP=\frac{AC^2+AB^2-CB^2}{2AB}=\frac{\frac{bbyy}{cc}+yy-\frac{bbyy}{aa}}{2y}=$$

$\frac{bby}{2cc}+\frac{1}{2}y-\frac{bbyy}{2aa}$; whence $\frac{bby}{2cc}+\frac{1}{2}y-\frac{bbyy}{2aa}=\sqrt{\frac{bbyy}{cc}-bb}$. That is, $aabby+aaccy-bbccy=2aac\sqrt{bbyy-bbcc}$; put $aabb+aacc-bbcc=d$, then $dy=2aac\sqrt{bbyy-bbcc}$; and by reduction,

$$y=\frac{aabcc}{\sqrt{a^4bbcc-\frac{1}{4}dd}}.$$

Fig.

PROB. LXXXVII.

32. *In the triangle* ABC, *there is given the rectangle of the ſides; the rectangle of the ſegments of the baſe, made by a perpendicular; and the area: to find the reſt.*

Let the area $= b$, AD × DC $= c$, AB×BC$=d$, and BD$=z$, $2y=$difference of the ſegments AD,DC. Then $\frac{2b}{z} =$ AC, and $\frac{b}{z}+y=$ DC, $\frac{b}{z}-y=$ DA. Whence $\frac{bb}{zz}-yy=c$, and $\sqrt{z^2+\overline{\frac{b}{z}+y}\big|^2}\times\sqrt{z^2+\overline{\frac{b}{e}-y}\big|^2}=d$; and ſquaring all the quantities, and putting $\frac{bb}{zz}-c$ for yy, and v for $zz+\frac{2bb}{zz}-c$, and then $\overline{v+\frac{2by}{z}}\times\overline{v-\frac{2by}{z}} = dd = vv - \frac{4bbyy}{zz} =$ (reſtoring the values of v and y) $z^4+\frac{4b}{z^4}$ $+cc+4bb-2czz-\frac{4cbb}{zz}-\frac{4b^4}{z^4}+\frac{4bbc}{zz}$, or $z^4 - 2czz+4bb+cc=dd$. Whence z is known, an $y=\sqrt{\frac{bb}{zz}-c}$, and then AD, DC, and AB, BC will be found.

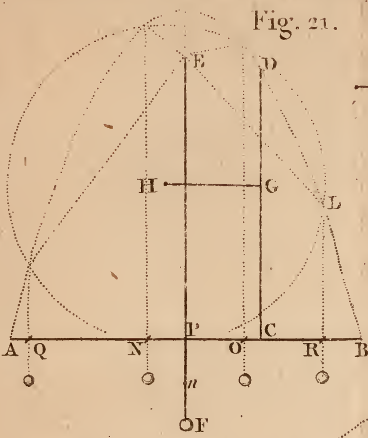
PROB. LXXXVIII.

33. *In the right-angled triangle* ABD, *there is given the perpendicular on the hypothenuſe; and the radius of the inſcribed circle: to find the ſides.*

Put the perpendicular BQ $=p$, radius CR $=r$ AD$=a$, AB$=e$, BD $=y$. Then (Geom. II. 21.

$aa =$

Fig. 21.



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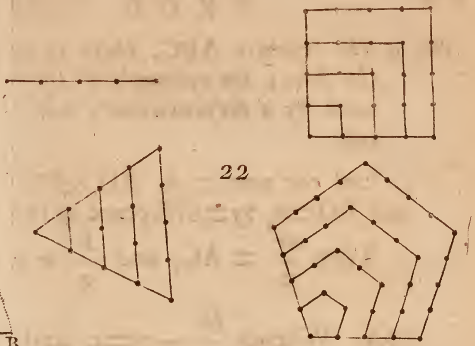

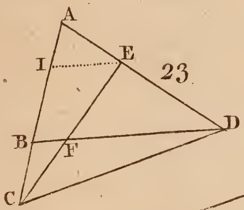

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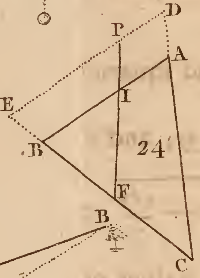

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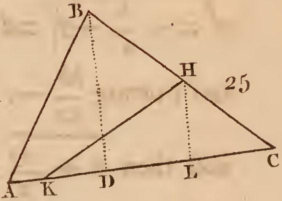

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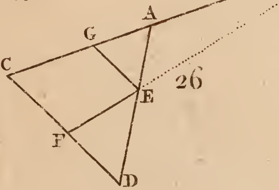

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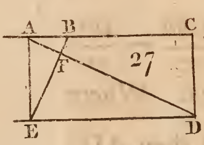

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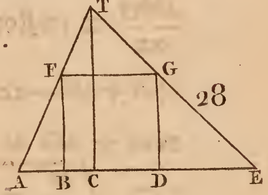

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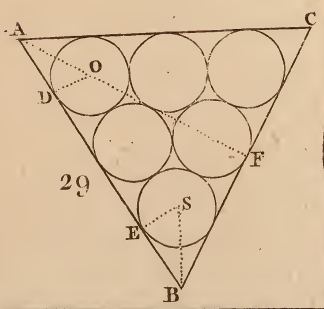

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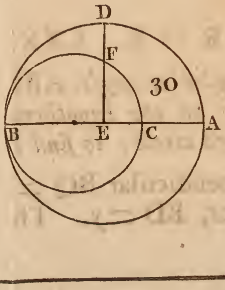

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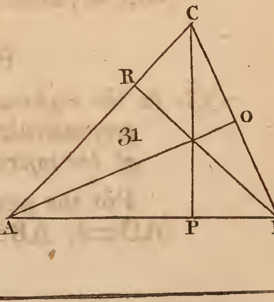


31

Pl. IV. *pe 390*

Fig. 33. $aa = ee + yy$. And (II. 20. cor. 2.) $pa = ey$. But AD or AF+FD=AR+DI, and AD+2CR = AB+BD, that is, $\underline{a+2r} = \underline{e+y}$; whence $aa+2pa = ee + 2ey + yy = \overline{e+y}|^2 = \overline{a+2r}|^2 = aa+4ra+4rr$, therefore $2pa - 4ra = 4rr$, and $a = \frac{2rr}{p-2r}$. Alſo $aa - 2pa = ee - 2ey + yy = \overline{e-y}|^2$, and $e - y = \sqrt{aa - 2pa}$. Whence $e = \frac{a + 2r + \sqrt{aa - 2pa}}{2}$, and $y = \frac{a + 2r - \sqrt{aa - 2pa}}{2}$.

PROB. LXXXIX.

34. *There is an iſoceles triangle, in which two circles are inſcribed, touching one another and the ſides of the triangle; their diameters are* 8 *and* 12: *to find the ſides of the triangle.*

From the centers D, F, draw DG, FH ⊥ to BC, and FO ∥ to CB . . draw CFDA.

Put DG $= r$, FH $= s$, DO $= r - s = c$, FD $= r + s = b$. Then FO $= \sqrt{bb - cc} = d$, and CB $= a$.

The triangles DFE and BCA are ſimilar, whence $b : d :: a : \frac{da}{b} =$ AC, and $c : b :: r : \frac{rb}{c} =$ CD, then $r + \frac{rb}{c} =$ AC $= \frac{da}{b}$, whence $a = \frac{bbr}{cd} + \frac{br}{d} = \frac{360}{\sqrt{96}}$.

Fig.

PROB. XC.

35. *There is given* AD, *and* CD *the radius of the ſemicircle* CEG, *to find the radius of a circle inſcribed between* AC, *the tangent* AE, *and the circle* CE.

Draw from the center O, the line OI perpendicular to AC; through O, draw AOF biſecting the angle DAE, and put radius DE $= r$, AD $= d$, OI $= a$, then AE $= \sqrt{dd - rr} = b$.

Then (Geom. II. 25.) AD : AE : : DF : EF, and AD+AE : AE : : DE : EF; that is, $d + b : b :: r : \frac{br}{d+b} =$ EF, DO $= a + r$, and DI $= \sqrt{DO^2 - OI^2} = \sqrt{rr + 2ra}$, and AI $= d - \sqrt{rr + 2ra}$, and by the ſimilar triangles AEF and AOI, $d - \sqrt{rr + 2ra} : a :: b : \frac{br}{b+d}$. Then $ba = \frac{dbr}{b+d} - \frac{br}{b+d}\sqrt{rr + 2ra}$. Put $b + d = c$, and reducing,

$$\begin{array}{lll} ccaa & -\ 2r^3a & =\ r^4 \\ & -\ 2drc & -\ ddrr. \end{array}$$

PROB. XCI.

36. *Through a given point* B, *to draw the right line* BDC, *ſo that the part* DC *comprehended between the two lines* AC, AH, *equidiſtant from* B, *may be of a given length.*

Produce CA to E, and compleat the rhombus EABH; make the angle CDF = CAF, and let CD $= a$, AE or AH $= b$, BA $= d$, AC $= x$, AF $= y$.

The triangles CAD, CEB are ſimilar, therefore CA (x) : CD (a) : : AE (b) : DB $= \frac{ab}{x}$. Since $\angle$ FDC = FAC, therefore their ſupplements

FDB = CAB, and ſo the triangles BAC and BDF are ſimilar, whence BA (d) : AC (x) : : DB $\left(\frac{ab}{x}\right)$: DF $= \frac{ab}{d}$. Fig. 36.

But the triangles FAD and FDB are ſimilar; for $\angle$BDF = FAD, (for BAD = BAE = FAC, add DAC, then CAB = FAD, that is, FDB = FAD;) and $\angle$F is common; therefore AF (y) : DF $\left(\frac{ab}{d}\right)$: : DF : $\frac{aabb}{ddy}$ = FB $= d+y$, which reduced is $ddyy + d^3y = aabb$; whence y will be had.

Again, the triangles DAF and BAC are alſo ſimilar, and CB $= a + \frac{ab}{x}$, then DF $\left(\frac{ab}{d}\right)$: AF (y) : : CB : CA (x); whence $\frac{abx}{d} = \frac{aby}{x} + ay$, reduced $abxx - adyx = abdy$, whence x is had. Then CE ($b + x$) : EB (b) : : AC (x) : AD $= \frac{bx}{b+x}$.

PROB. XCII.

Through a given point B, *to draw the right line* BDC, *ſo that the part* DC, *included between the lines* AC, AH, *may be given.* 36.

Through B, draw BH, BE parallel to EAC and AH, and put CD $= a$, AE $= b$, AH $= c$, AB $= d$, PH $= f$, BP being perpendicular to AH, and AC $= x$.

The triangles CAD and CEB are ſimilar, and CE ($x + b$) : EB (c) : : CA (x) : AD $= \frac{cx}{b+x}$; and DH $= c - \frac{cx}{b+x} = \frac{bc}{b+x}$. And (Geom. II. 22.) BD $= \sqrt{bb + \frac{ccbb}{bb+2bx+xx} + \frac{2bcf}{b+x}}$, and

b :

Fig. 36. $b : \sqrt{bb + \frac{ccbb}{bb+2bx+xx} + \frac{2bcf}{b+x}} :: x : a$;

whence $bbaa = bbxx + \frac{ccbbxx}{bb+2bx+xx} + \frac{2bcfxx}{b+x}$,

which reduced is
$$\begin{array}{l} x^4+2bx^3+bbx^2-2baax-bbaa=0. \\ \qquad\qquad\quad +\frac{2cf}{b}\quad +2cf \\ \qquad\qquad\quad -aa \end{array}$$

PROB. XCIII.

37. *The difference of the height of two hills being given, and their diſtance; to find their heights.*

Let BA, ED be the hills, put radius r = CR = 698000, DE—BA $= b = 119$, AB $= a$, BE $= c = 63$. Then CB $= r+a$, CE $= r+b+a$. Then (Geom. II. 21.) BR $= \sqrt{2ra+aa}$, RE $= \sqrt{bb+2br+2ra+2ba+aa}$, whence BE $= \sqrt{2ra+aa} + \sqrt{bb+2br+2ra+2ba+aa} = c$, and $\sqrt{bb+2br+2ra+2ba+aa} = c - \sqrt{2ra+aa}$, and by ſquaring, $bb+2br+2ra+2ba+aa = cc+2ra+aa-2c\sqrt{2ra+aa}$, and $2c\sqrt{2ra+aa} = cc-bb-2rb-2ba = dd-2ba$ (by ſubſtitution); and when ſquared $8ccra+4ccaa = d^4-4ddba+4bbaa$, and when reduced, $\left.\begin{array}{r} aa+2ra \\ +b \end{array}\right\} = \frac{cc-bb-4rb}{4} + \frac{rrbb}{cc-bb}$, and $a = 164{,}69$.

PROB. XCIV.

38. *Three lines drawn from the three angles of a triangle to the middle of the oppoſite ſides, being given; to find the ſides.*

Put AD $= b = 18$, E $= c = 24$, BF $= d = 30$, CB $= x$, AB $= y$, AC $= z$.

Then (Geom. II. 28.) $yy+zz = 2bb+\frac{1}{2}xx$, $yy+xx = 2dd+\frac{1}{2}zz$, $zz+xx = 2cc+\frac{1}{2}yy$; and adding theſe

Fig. 38. theſe three equations, $2xx+2yy+2zz=2bb+2cc+2dd+\frac{1}{2}xx+\frac{1}{2}yy+\frac{1}{2}zz$, and $xx+yy+zz=\frac{4}{3}bb+\frac{4}{3}cc+\frac{4}{3}dd$, from this ſubtract the firſt equations, then $xx=\frac{4}{3}dd-\frac{2}{3}bb+\frac{4}{3}cc-\frac{1}{2}xx$, or $9xx=8cc+8dd-4bb$, $9yy=8bb+8dd-4cc$, $9zz=8bb+8cc-4dd$, whence $x=34{,}176$; $y=28{,}844$; $z=20$.

PROB. XCV.

39. ABC *is an equilateral triangle,* O *a point in it equidiſtant from* A, B, C. *If the ſides, and the line* BO *be all produced till they cut the line* PD *in* D, E, R, P; *then there is given* DE, ER, RP; *to find the ſide of the triangle* ABC, *and the area.*

Draw EF, EG parallel to BP, BR; and put $DE=a=304$; $ER=b=121.6$; $RP=c=159{,}6$; and $DR=d$, $DP=s$, CL or $AL=x$, CG or $FG=y$. Then (Geo. II. 39. cor.) $BL=x\sqrt{3}$, $EG=y\sqrt{3}$.

The triangles DEF and DPA are ſimilar, whence $a:2y::s:\frac{2sy}{a}=AP$, and $PB=2x+\frac{2sy}{a}$.

Since $\angle PBR=\angle EBR$, therefore (Geo. II. 25.) $2x+\frac{2sy}{a}:2x+2y::c:b$, and $2bx+\frac{2bcy}{a}=2cx+2cy$, whence $cx-bx=\frac{bsy}{a}-cy$, and $y=\frac{ca-ba}{bs-ac}x=\frac{fx}{g}$, by ſubſtitution.

Again DE (a) : EG $(y\sqrt{3})$ :: DR (d) : $\frac{dy}{a}\sqrt{3}=RL=\frac{dfx}{ag}\sqrt{3}$, and $RB=x\sqrt{3}+\frac{dfx}{ag}\sqrt{3}$, and $PB=2x+\frac{2sfx}{ag}$, and $BE=2x+2y=2x+\frac{2fx}{g}$.

But

Fig. 39. But (Geom. II. 26.) $BR^2 + PR \times RE = PB \times EB$, that is, $3xx \times \overline{1 + \frac{df}{ag}}\Big|^2 + bc = 4xx \times \overline{1 + \frac{f}{g}} \times \overline{1 + \frac{sf}{ag}}$: And by reduction, $1 + \frac{4f}{g} + \frac{4sf - 6df}{ag} + \frac{4saff - 3ddff}{aagg} \times$ into $xx = bc$. Whence $x = 78.4$, $y = 40$, and the area ABC $= 10646.16$.

PROB. XCVI.

40. *In the triangle* ABC, *there is given the baſe, and difference of the ſides, and the area: to find the triangle.*

Let the area $= f = 796$; difference of the ſides CA, CB $= b = 10$; baſe AB $= d = 50$; perpendicular CD $= \frac{2f}{d} = p = 31.84$, and AD $= a$. Then AC $= \sqrt{aa + pp}$, and CB $= \sqrt{\overline{d - a}|^2 + pp}$; therefore by the queſtion $\sqrt{aa + pp} + b = \sqrt{dd - 2da + aa + pp}$, which ſquared is $aa + bb + pp + 2b\sqrt{aa + pp} = dd - 2da + aa + pp$, and $2b\sqrt{aa + pp} = dd - bb - 2da$, and ſquaring both ſides $4bbaa + 4bbpp = d^4 + b^4 - 2ddbb - 4d^3a + 4bbda + 4ddaa$. Which reduced is $aa - da = \frac{bbpp}{dd - bb} - \frac{dd - bb}{4}$, whence $a = 16.739$, AC $= 36$, BC $= 46$, BD $= 33.261$.

PROB. XCVII.

41. *There is given the ſide of a rhombus, and the ſide of its inſcribed ſquare; to find the area.*

Let AB $=$ BD $= d = 4\frac{3}{8}$, CO $=$ CE $= s = 3$, BC $= x$. Then DC $= d - x$, and AC $= d + x$.

The triangles ACE and CDO are ſimilar, and Fig. 41. $d+x : s :: d-x : \frac{d-x}{d+x}s = DO$. And (Geom. II. 21.) $ss + ss \times \frac{\overline{d-x}|^2}{\overline{d+x}|^2} = \overline{d-x}|^2$; that is $2ss \times \overline{dd+xx}$ $= \overline{dd-xx}|^2$; reduced, $\left.\begin{array}{lll} x^4 & -ddxx & +d^4 \\ & -2ss & -2ssdd \end{array}\right\} = 0$. Whence $x = \sqrt{ss+dd \pm \sqrt{ss+4ad}} = \frac{5}{8}$, and AC$=5$, AE$=4$, DO$=2\frac{1}{4}$, DQ$=5\frac{1}{4}$, area$=73\frac{1}{2}$, QA $= 7$.

PROB. XCVIII.

Given the four ſides of a trapezium, inſcribed in a circle; to find the diagonals, and diameter of the circle. 42.

Let AB$=a$, BC$=b$, CD$=c$, AD$=d$, BE$=x$, the triangles ABE, and CED are ſimilar; for ∠ABE = ECD (Geom. IV. 12. cor. 2.); and the angles at E are vertical; therefore AB (a) : BE (x) :: DC (c) : CE $= \frac{cx}{a}$; alſo the triangles AED and BEC are ſimilar, and BC (b) : CE $\left(\frac{cx}{a}\right)$ :: AD (d) : DE $= \frac{dcx}{ab}$. And BC (b) : BE (x) :: AD (d) : AE $= \frac{dx}{b}$. Then BD $= x + \frac{dcx}{ab}$, and AC $= \frac{dx}{b} + \frac{cx}{a}$. Then (Geom. IV. 32.) AC $\times$BD$=$AB$\times$CD$+$AD$\times$BC, or $\frac{dxx}{b} + \frac{ddcxx}{abb} + \frac{cxx}{a} + \frac{dccxx}{baa} = ac + bd$, whence x is had, and then AC and BD are known.

Then

Fig. 42. Then ſuppoſe a perpendicular from A upon BD; then (Geom. II. cor. 23.) the diſtance of the perpendicular from D is $= \frac{AD^2 + DB^2 - AB^2}{2BD} = f$.

And $\sqrt{AD^2 - ff} =$ the perpendicular $= p$; and (Geom. IV. 28.) $p : AD :: AB :$ diameter of the circumſcribing circle $= \frac{AD \times AB}{p}$.

PROB. XCIX.

43. *The three ſemicircles* HFG, HEI, *and* GOI *touch one another in* H, G, *and* I; *to draw a fourth circle* FOE *to touch all the reſt.*

From the centers A, B, C draw the lines ADE, BD, and CD; and DP perpendicular to AC, and let $AG = a$, BE or $BI = b$, $CG = c$, and $DE = x$. Then $AD = a + x$, $BD = b - x$, $CD = c + x$, $AC = a + c$, $BC = b - c$.

In the triangle ADC (Geom. II. 22. cor.) $PC = \frac{\overline{c+a}^2 + \overline{c+x}^2 - \overline{a+x}^2}{2c + 2a}$, and in the triangle BDC, $PC = \frac{\overline{b-c}^2 + \overline{c+x}^2 - \overline{b-x}^2}{2b - 2c}$.

Whence

$$\frac{\begin{array}{l} cc + 2ca + aa \\ + cc + 2cx + xx \\ - aa - 2ax - xx \end{array}}{c + a} = \frac{\begin{array}{l} bb - 2bc + cc \\ + cc + 2cx + xx \\ - bb + 2bx - xx \end{array}}{b - c}$$

That is,

$$\frac{2cc + 2ca - 2cx - 2ax}{c + a} = \frac{2cc - 2bc + 2cx + 2bx}{b - c};$$

And

Fig. 43.

And multiplying alternately,

$$\begin{array}{ll} 2bcc + 2bca + 2bcx - 2abx = & 2c^3 - 2bcc + 2ccx + 2bcx \\ - 2c^3 - 2acc - 2ccx + 2cax & + 2cca - 2bca + 2cax \\ & + 2bax \end{array}$$

And

$$4abx + 4ccx = 4bcc + 4bca - 4c^3 - 4cca,$$

Whence $x = \frac{\overline{a+c} \times \overline{b-c}}{ab+cc} c.$

PROB. C.

In the triangle ACB, *there is given the ſides* AC, CB; *and the length and breadth of the inſcribed rectangular parallelogram* DEHF; *to find the reſt.* 44.

Draw CP perpendicular to AB, and let CA $= b$, CB $= c$, DE or GP $= p$, DF $= a$, CP $= z$, AB $= y$; and let $p = b + c$, $q = b - c$.

The triangles CDF and CAB are ſimilar, and $z : y :: z - p : a$; whence $za = zy - py$, and $zy - za = py$; therefore $z = \frac{py}{y-a}$.

Again, (Geom. II. 24.) $y : p :: q : \frac{pq}{y} =$ diff. ſegments of the baſe. Therefore AP $= \frac{1}{2}y + \frac{pq}{2g}$.

But (Geom. II. 21.) $bb = zz + \overline{\frac{1}{2}y + \frac{pq}{2y}}\Big|^2 = \frac{ppyy}{\overline{y-a}\,|^2} + \frac{1}{4}yy + \frac{1}{2}pg + \frac{ppqq}{4yy}$. Which equation reduced is

$$\begin{array}{lllll} y^6 - 2ay^5 & + aay^4 & + 8abby^3 & + ppqqy^2 & - 2ap^2q^2y + aappqq \\ & + 4pp & - 2apq & + 2aapq & = 0. \\ & + 2pq & & - 4aabb & \\ & - 4bb & & & \end{array}$$

PROB.

Fig.

PROB. CI.

45. *In the right-angled triangle* AVP, KD *is drawn parallel to the baſe, and there is given the baſe* AP, *and the ſegments* VD, AK; *to find the reſt.*

Let $AK=b=200$, $AP=c=400$, $VD=d=260$, and $DP=a$; then (Geom. II. 12.) $a:b::d:\frac{bd}{a}=VK$, and $VA=b+\frac{bd}{a}=\frac{b}{a}\times\overline{a+d}$. But $AV^2-VP^2=AP^2$, or $\frac{bb}{aa}\times\overline{a+d}|^2-\overline{a+d}|^2=cc$, or $\overline{bb-aa}\times\overline{a+d}|^2=aacc$, that is,

$$a^4+2da^3\begin{array}{l}+cc\\+dd\\-bb\end{array}aa-2bbdda=bbdd.$$

And $a=141{,}727$, and $a+d$ or $VP=401.727$.

PROB. CII.

46. *In the figure* CAFD; CA, BP, DF *are perpendicular to* AF; *and the ſides of the triangle being produced, there is given* HA, AC, CB, *and* BD, DF, FT; *to find the ſides of the triangle* HBT.

Let $HA=n$, $AC=p$, $CB=f$, $BD=d$, $DF=c$, $FT=b$, and $TH=a$, $TP=z$, $PH=v$, $BP=x$, $BT=y$, $e=HB$.

The triangles TBP and TCA are ſimilar, and $y:x::f+y:p$, and $py=fx+yx$, or $py-xy=fx$, whence $y=\frac{fx}{p-x}$. The triangles HBP and HDF are ſimilar, and $e:x::e+d:c$, or $ce=cx+xd$, and $ce-xe=dx$, whence $e=\frac{dx}{c-x}$. Likewiſe $z:x::$

$a+n : p$, and $pz = \overline{a+n} \times x$, and $z = \frac{a+n}{p}x$. Fig. 46.
Likewiſe $v : x :: a+b : c$, and $cv = \overline{a+b} \times x$, and $v = \frac{a+b}{c}x$.

But $v+z=a=\frac{a+n}{p}x + \frac{a+b}{c}x$, whence $pca = cax+cnx+pax+pbx$, and $pca-cxa-pxa=cnx+pbx$; therefore $a = \frac{cnx+pbx}{pc-cx-px}$.

Again, $TC = f + \frac{fx}{p-x} = \frac{pf}{p-x}$, and (Geom. II. 21.) $\sqrt{\frac{ppff}{\overline{p-x}|^2} - pp} = n+a = n + \frac{cnx+pbx}{pc-cx-px} = \frac{pcn-ncx-npx+ncx+pbx}{pc-cx-px} = \frac{pcn+rpx}{pc-sx}$, (putting $r=b-n$, and $s=c+p$); and by ſquaring, $\frac{ppff-p^4+2p^3x-ppxx}{pp-2px+xx} = \frac{ppccnn+2ppcnrx+rrppxx}{ppcc-2pcsx+ssxx}$, which reduced is

$$\begin{array}{lllll}
pprrx^4 & +2ppcnrx^3 & +ppccnnx^2 & -2p^3ccnnx & +p^4ccnn=0 \\
+ppss & -2p^3rr & -4p^3cnr & +2p^4cnr & +p^6cc \\
 & -2p^3ss & +p^4rr & -2cp^5s & -ccp^4ff \\
 & -2p^3cs & +p^4ss & +2cp^3ffs & \\
 & & -ppffss & -2ccp^5 & \\
 & & +4p^4cs & & \\
 & & +p^4cc & &
\end{array}$$

PROB. CIII.

Given the ſides and area of a trapezium; to find the diagonal. 47.

Draw the perpendiculars BE, DF upon the diagonal AC; and put AB$=a=4$, BC$=b=6$, CD$=c=7$, DA$=d=5$, and $f=$the area, and AC$=y$.

Fig. 47. Then (Geom. II. 23. cor.) CE $= \frac{yy + bb - aa}{2y}$, and BE $= \sqrt{bb - \overline{\frac{yy + bb - aa}{2y}}|^2} =$ $\sqrt{\frac{4bbyy - y^4 - 2yy \times \overline{bb - aa} - \overline{bb - aa}|^2}{4yy}} =$ $\sqrt{\frac{-y^4 + yy \times \overline{2bb + 2aa} - \overline{bb - aa}|^2}{4yy}}$. In like manner DF $= \sqrt{\frac{-y^4 + yy \times \overline{2cc + 2dd} - \overline{cc - dd}|^2}{4yy}}$.

Put $\overline{2aa + 2bb} \times yy - \overline{bb - aa}|^2 = pyy - qq = v$.
and $\overline{2cc + 2dd} \times yy - \overline{cc - dd}|^2 = ryy - ss = z$.
Then $\frac{1}{2} y \times \overline{\mathrm{BE} + \mathrm{DF}} = f$, and $2y \times \overline{\mathrm{BE} + \mathrm{DF}} = 4f$, that is, $\sqrt{v - y^4} + \sqrt{z - y^4} = 4f$, and by squaring $v + z - 2y^4 + 2\sqrt{vz \,{}^{-v}_{-z} y^4 + y^8} = 16ff$, and $\sqrt{vz \,{}^{-v}_{-z} y^4 + y^8} = 8ff - \frac{v + z}{2} + y^4$, and by squaring, $vz \,{}^{-v}_{-z} y^4 + y^8 = 64f^4 - 8ff \times \overline{v + z} + \frac{\overline{v + z}|^2}{4} + \overline{16ff - v - z} \times y^4 + y^8$. And $vz = 64f^4 - 8ff \times \overline{v + z} + \frac{\overline{v + z}|^2}{4} + 16ffy^4 = 64f^4 - 8ff \times \overline{v + z} + \frac{vv}{4} + \frac{2vz}{4} + \frac{zz}{4} + 16ffy^4$, or $16ffy^4 - 8ff \times \overline{v + z} + \frac{vv - 2vz + zz}{4} + 64f^4 = 0$; that is, $64ffy^4 - 32ff \times \overline{v + z} + \overline{v - z}|^2 + 256f^4 = 0$, and restoring the values of v and z, we ſhall have

64ff

Fig. 47.

$$\left.\begin{array}{l} 64ff \\ +\overline{p-r}|^2 \end{array}\right\} y^4 \left.\begin{array}{l} +\overline{2p-2r}\times\overline{ss-qq} \\ -32ff \quad +\overline{p+r} \end{array}\right\} y^2 \left.\begin{array}{l} +\overline{ss-qq}^2 \\ +32ff\times\overline{ss+qq} \\ +256f^4 \end{array}\right\} = 0.$$

and $y = 7.68$.

PROB. CIV.

In the right-angled triangle DCF, *there is given* DC+CF, *and* BA *parallel to the baſe, and* DA; *to find the reſt.* 48.

Let $DC + CF = s$, $BA = b$, $DA = d$, $CF = a$; then $CD = s - a$, $DF = \sqrt{DC^2 - CF^2} = \sqrt{ss - 2sa}$. The triangles CAB and CFD are ſimilar, whence $\sqrt{ss - 2sa} : a :: b : \frac{ab}{\sqrt{ss - 2sa}} = CA$; and $\sqrt{DA^2 - DF^2} = \sqrt{dd - ss + 2sa} = AF$; whence $\frac{ba}{\sqrt{ss - 2sa}} + \sqrt{dd - ss + 2sa} = a$, and multiplying by $\sqrt{ss - 2sa}$, and tranſpoſing, $\sqrt{ddss - 2ddsa - s^4 + 4s^3a - 4ssaa} = \sqrt{ssaa - 2sa^3} - ba$; and ſquaring,

$$\begin{array}{l} -4ssaa + 4s^3a + ddss \\ \qquad\quad -2ddsa - s^4 \end{array} = \begin{array}{l} -2sa^3 + ssaa \\ \qquad\quad + bbaa \end{array} - \sqrt{4bbssa^4 - 8bbsa^5};$$

and tranſpoſing again,

$$\sqrt{4bbssa^4 - 8bbsa^5} = \begin{array}{l} -2sa^3 + 5ssaa - 4s^3a + s^4 \\ \qquad\qquad + bb \quad + 2dds - ddss \end{array}$$

$$= -ca^3 + faa + ga + h,$$ by ſubſtitution, and by ſquaring,

$$4bbssa^4 - 8bbsa^5 = cca^6 - 2cfa^5 + \begin{array}{l} ffa^4 - 2cha^3 + gga^2 + 2gha + hh. \\ -2cg \quad + 2fg \quad + 2fh \end{array}$$

Fig.

PROB. CV.

49. *There are given the three ſides of the triangle* ABC, *and the angles* A, *and* B, *are biſected by the lines* AD, BE; *to find the length of one as* AD, *and alſo the diſtance* AF *to the point of interſection* F.

Put AB$=a$, BC$=b$, AC$=c$, and AD$=x$, AF$=y$. Then (Geom. II. 25.) AB : AC : : BD : DC, and AB+AC : AB : : BD+DC : BD; that is, $a+c : a :: b : \frac{ab}{a+c}=$ BD; likewiſe $a+c ; c :: b : \frac{bc}{a+c}$ $=$CD. But (Geom. II. 26.) AD^2+BDC$=$BAC that is, $xx+\frac{abbc}{\overline{a+c}|^2}=ac$, and $xx=ac-\frac{bb}{\overline{a+c}|^2}\times$ $ac=ac\times\frac{\overline{a+c}|^2-bb}{\overline{a+c}|^2}=ac\times\frac{\overline{a+c+b}\times\overline{a+c-b}}{\overline{a+c}|^2}$, whence $x=\frac{\sqrt{ac\times\overline{a+c+b}\times\overline{a+c-b}}}{a+c}=$AD.

Again, AB + BD : AD : : AB : AF, that is; $a+\frac{ab}{a+c}:\frac{\sqrt{ac\times\overline{a+c+b}\times\overline{a+c-b}}}{a+c}::a:$ $y=\frac{a\sqrt{ac\times\overline{a+c+b}\times\overline{a+c-b}}}{\overline{a+c}\times\overline{a+\frac{ab}{a+c}}}=$ $\frac{\sqrt{ac\times\overline{a+c+b}\times\overline{a+c-b}}}{a+c+b}$, that is,

$$y=\sqrt{\frac{ac\times\overline{a+c-b}}{a+c+b}}=\text{AF}.$$

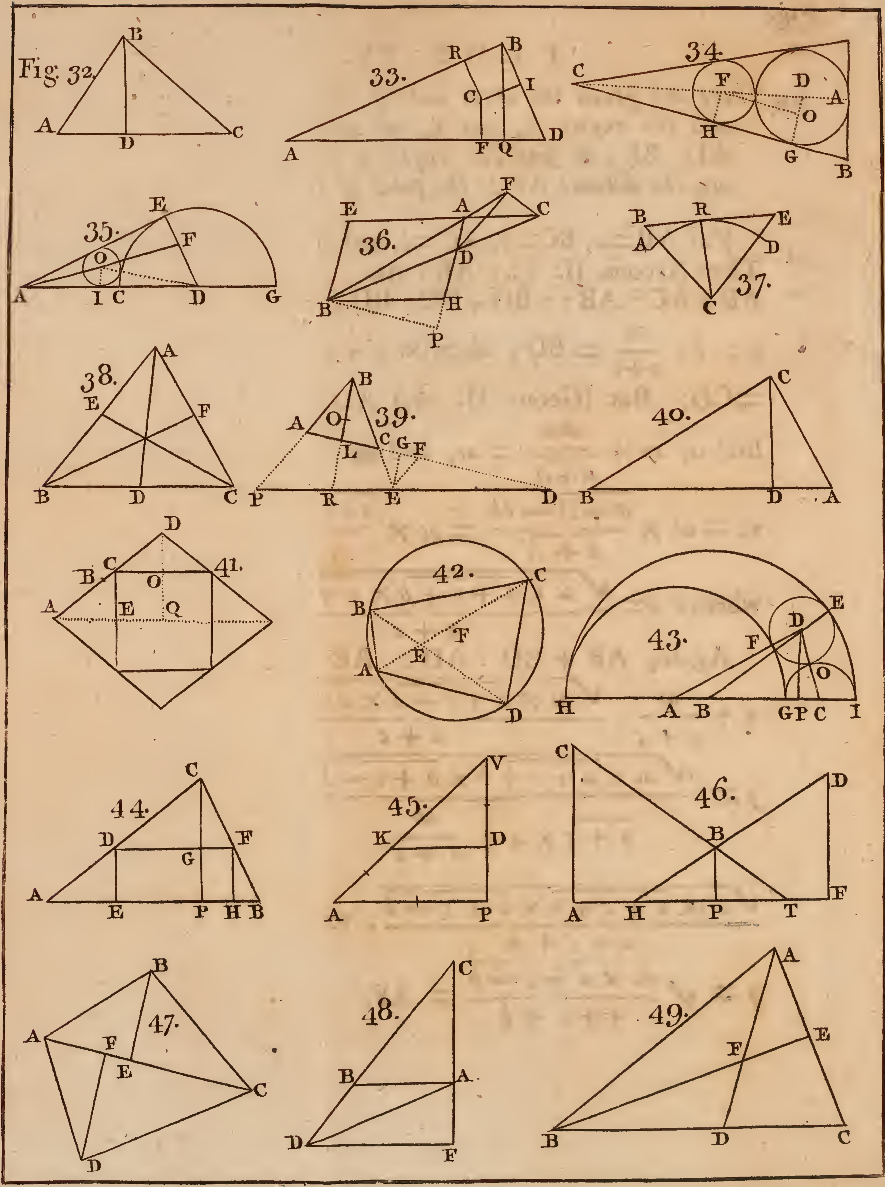


Fig. 32.
A B C D
33.
A B C D F Q R I
34.
C F D A O H G B
35.
A E F O I C D G
36.
E A F C D B H P
37.
B R E A D C
38.
A E F B D C
39.
B A O L C G F P R E D
40.
C B D A
41.
D C B O A E Q
42.
B C F E A D
43.
H A B F D E O G P C I
44.
C D F G A E P H B
45.
V K D A P
46.
C D B A H P T F
47.
B A F E C D
48.
C B A D F
49.
A E F B D C

Pl.V. *pa*.404

Fig.

PROB. CVI.

The diameters of three circles being given, which are deſcribed from the angular points of a triangle, as centers, whoſe three ſides are given; to find the radius of a fourth circle to touch all the three. 50.

Let ABC be the given triangle, D the center of the circle required; on AB let fall the perpendiculars DE, CK, and draw DF perpendicular to AC. And put AB $= b$, AC $= c$, CB $= d$, and AO $= r$, BR $= s$, CT $= t$; and AK $= g$, KC $= h$; and AE $= x$, AF $= y$, OD $= a$. In the triangle ADB, (Geom. II. 23.) $aa+2as+ss=aa+2ar+rr+bb-2bx$. Whence $2bx=rr+bb-ss-2as+2ar$, and $x=\frac{rr+bb-ss-2as+2ar}{2b}$. And in the triangle ADC, $aa+2at+tt=aa+2ar+rr+cc-2cy$, and $2cy=rr+cc-tt-2ta+2ra$, and $y=\frac{rr+cc-tt-2ta+2ra}{2c}$.

The triangles ACK, AFG are ſimilar, and $g:c::y:\frac{cy}{g}=$ AG, then $x-\frac{cy}{g}=$ GE. Alſo the triangles DGE (AGF) and ACK are ſimilar; whence $h:g::x-\frac{cy}{g}:\sqrt{aa+2ar+rr-xx}=$ DE. Whence $h\sqrt{aa+2ar+rr-xx}=gx-cy$.

Put $l=rr+bb-ss$, $f=2s-2r$, $m=rr+cc-tt$, $n=2t-2r$, $p=lg-bm$, $q=bn-fg$. Then $x=\frac{l-fa}{2b}$, and $b\sqrt{aa+2ar+rr-\overline{\frac{l-fa}{2b}}\Big|^2}=\frac{lg-fga}{2b}-\frac{m-na}{2}=\frac{p+qa}{2b}$. Which ſquared is $bhaa+2rbha$

Fig. 50. $$+rrbb-bb\times\frac{ll-2lfa+ffaa}{4bb}=\frac{pp+2pqa+qqaa}{4bb}$$

and reduced

$$\begin{array}{llll} 4bbbbaa & +8bbbbra & +4bbbbrr & =0. \\ -qq & -2dfbb & -bbll & \\ -ffbb & -2pq & -pp & \end{array}$$

PROB. CVII.

51. *To find the point* D, *from which three lines* DA, DB, DC *drawn to the three given points* A, B, C; *ſhall have a given ratio.*

Draw AC, and DFG, BE perpendicular to it. Draw BG ∥ to EF; and put AE $=a$, AC $=b$, EB $=c$; and AF $=x$, FD $=y$. Then CF $=b-x$, FE $=x-a$, and let DA, DB, DC, be as 1, r, and s.

Then (Geom. II. 21.) AD$^2=xx+yy$; BD$^2=\overline{c+y}|^2+\overline{x-a}|^2=cc+2cy+yy+xx-2xa+aa$; and CD$^2=bb-2bx+xx+yy$.

But by the queſtion $1:rr::$ DA^2 : DB$^2=rr\times$DA^2, and $1:ss::$ DA^2 : DC$^2=ss\times$DA^2; that is, $cc+2cy+yy+xx-2xa+aa=rrxx+rryy$, and $bb-2bx+xx+yy=ssxx+ssyy$, and putting $m=rr-1$, $p=cc+aa$, $f=ss-1$, we ſhall have theſe two equations,

$$\begin{array}{lll} (a) & (b) & (c) \\ myy & -2cy & +mxx=0. \\ & & +2ax \\ & & -p \end{array}$$

and

$$\begin{array}{lll} (f) & (g) & (h) \\ fyy & * & +fxx=0. \\ & & +2bx \\ & & -bb \end{array}$$

Then to expunge y (by Prob. liv. rule 2.) we have Fig. 52.

$$A = -2cf,\quad B = -2cfxx - 4bcx + 2cbb.$$

$$\begin{array}{l} D = mfxx + 2afx - pf \\ \quad - mfxx - 2mbx - mbb. \end{array}$$

Whence $AB + DD = 0$, that is,

$$\left.\begin{array}{l} \,4ccffxx + 8bccfx - 4bbccf \\ + 4aaff \quad - 4apff \quad + ppff \\ - 4ambf \quad - 4ambb + 2mbbpf \\ + 4mmbb + 4mbpf + mmb^4 \\ \qquad\quad + 4mmb^3 \end{array}\right\} = 0.$$

PROB. CVIII.

In a triangle, there is given a perpendicular, the difference of the ſides, and the difference of the ſegments of the baſe; to find the ſides. 53.

Let the perpendicular $CD = a$, $CB - CA = c$, and $BD - DA = b$, $DA = x$. Then $CA = \sqrt{aa + xx}$, and $CB = \sqrt{aa + xx} + c$, and $AB = 2x + b$. Then (Geom. II. 24.), $b + 2x : 2\sqrt{aa + xx} + c :: c : b$; whence $bb + 2bx = 2c\sqrt{aa + xx} + cc$, and $2bx + bb - cc = 2\sqrt{aa + xx}$, and ſquared is $4bbxx + 4bx \times \overline{bb - cc} + \overline{bb - cc}^2 = \overline{4aa + 4xx} \times cc$;

reduced

$$\begin{array}{llll} 4bbxx & + 4b^3x & = & 4aacc. \\ -4cc & - 4bcc & - & b^4 \\ & & + & 2bbcc \\ & & - & c^4 \end{array}$$

Fig.

PROB. CIX.

54. *There is given the perpendicular in a triangle, and the two differences between the leaſt ſide, and the other two; to find the ſides.*

Let the perpendicular $AD = a$, $BC - BA = b$, $AC - AB = c$, $AB = x$; then $BC = b + x$, $AC = c + x$, and $BD = \sqrt{xx - aa}$, and $DC = b + x - \sqrt{xx - aa}$; and (Geom. II. 24.) $BC\ (b+x) : AC + AB\ (c + 2x) :: AC - AB\ (c) : DC - DB\ (b + x - 2\sqrt{xx - aa})$; whence $\overline{b+x}|^2 - \overline{2b+2x} \times \sqrt{xx-aa} = cc + 2cx$; and $\overline{2b + 2x} \times \sqrt{xx - aa} = \begin{array}{l} bb + 2bx + xx \\ - cc - 2cx \end{array}$

Put $bb - cc = d$, $2b - 2c = f$; then $\overline{4bb + 8bx + 4xx} \times \overline{xx - aa} = \overline{xx + fx + d}|^2$, which multiplied and reduced is

$$\begin{array}{llllll} 3x^4 & + 8bx^3 & + 4bbx^2 & - 8baax & - 4bbaa & = 0. \\ & - 2f & - 4aa & - 2df & - dd & \\ & & - ff & & & \\ & & - 2d & & & \end{array}$$

PROB. CX.

55. *Having all the ſides of a right-angled triangle* ACB; *to find either ſegment of the baſe* AD, *the perpendicular* CD, *the area, and the radius of the inſcribed circle,* &c.

1. Let $AC = a$, $CB = c$, $AB = b$, $z = \frac{a+b+c}{2}$; then (Geom. II. 20. cor. 1.) $aa = b \times AD$, and $AD = \frac{aa}{b}$. But $aa = bb - cc = \overline{b+c} \times \overline{b-c}$; therefore $AD = \frac{\overline{b+c} \times \overline{b-c}}{b}$, that is,

the ſegment AD $= \frac{aa}{b} = \frac{\overline{b+c}\times\overline{b-c}}{b}$. Fig. 55.

2. *For the perpendicular* CD.

$CD^2 = AC^2 - AD^2 = aa - \frac{a^4}{bb} = \frac{aa\times\overline{bb-aa}}{bb}$, and $CD = \frac{a}{b}\sqrt{bb-aa} = \frac{a}{b}\sqrt{\overline{b+a}\times\overline{b-a}}$; or $CD = \frac{a}{b}\sqrt{cc} = \frac{ac}{b}$. Therefore the perpendicular $CD = \frac{ac}{b} = \frac{a}{b}\sqrt{bb-aa} = \frac{a}{b}\sqrt{\overline{b+a}\times\overline{b-a}}$.

3. *For the area.*

$\frac{AB\times CD}{2}$ = area; that is, the area $= \frac{ac}{2} = \frac{a\sqrt{\overline{b+a}\times\overline{b-a}}}{2}$. And ſince $aa+cc=bb$, add $2ac$, then $aa+2ac+cc$ or $\overline{a+c}|^2 = bb+2ac$, and $2ac = \overline{a+c}|^2 - bb$, and $\frac{ac}{2}$ or the area $= \frac{\overline{a+c}|^2 - bb}{4} = \frac{\overline{a+c+b}\times\overline{a+c-b}}{4} = z\times\overline{z-b}$. And ſince $\overline{a+c}|^2 + \overline{a-c}|^2 = 2aa+2cc = 2bb$, therefore $\overline{a+c}|^2 - bb = bb - \overline{a-c}|^2$. Therefore the area $= \frac{bb-\overline{a-c}|^2}{4} = \frac{\overline{b+a-c}\times\overline{b-a+c}}{4} = \overline{z-a}\times\overline{z-c}$. Hence the area $= \frac{ac}{2} = \frac{a\sqrt{\overline{b+a}\times\overline{b-a}}}{2} = \frac{\overline{a+c+b}\times\overline{a+c-b}}{4}$ $= \frac{\overline{b+a-c}\times\overline{b-a+c}}{4} = z\times\overline{z-b} = \overline{z-a}\times\overline{z-c}$.

4 For

Fig. 55.

4. *For the radius of the inſcribed circle.*

The area $= \frac{a+b+c}{2} \times$ radius of the inſcribed circle (Geom. IV. 30. cor.) $= \frac{a+b+c}{2} \times r$, putting r for the radius; then $r = \frac{2 \text{ area}}{a+b+c} = \frac{a+c-b}{2} = z-b$; or the radius $= \frac{ac}{a+b+c}$, that is, the radius of the inſcribed circle is $= \frac{ac}{2z} = \frac{a+c-b}{2} = z-b$.

5. *For the circumſcribing circle.*

The radius of the circumſcribing circle $= \frac{1}{2} b$ (Geom. IV. 14.)

6. *For the tangents.*

The tangent, or the diſtance from A to the point of contact of the inſcribed circle $= \frac{a+b-c}{2} = z-c$, (Geom. IV. 30).

And the diſtance from the right angle C is $= r$ the radius.

7. *For the diſtance of the center.*

The diſtance from A $= \sqrt{rr + \left.\frac{\overline{a+b-c}}{2}\right|^2} = \sqrt{\left.\frac{\overline{a+c-b}}{2}\right|^2 + \left.\frac{\overline{a+b-c}}{2}\right|^2}$ (by Art. 4. and 6) $= \sqrt{\frac{\overline{a+\overline{b-c}}|^2}{4} + \frac{\overline{a-\overline{b-c}}|^2}{4}} = \sqrt{\frac{\overline{a+d}|^2}{4} + \frac{\overline{a-d}|^2}{4}}$

(putting $d=b-c$) $=\sqrt{\frac{2aa+2dd}{4}}=\sqrt{\frac{aa+\overline{b-c}|^2}{2}}$ Fig. 55.

$=\sqrt{\frac{aa+bb-2bc+cc}{2}}=\sqrt{\frac{2bb-2bc}{2}}=\sqrt{bb-bc}$

$=\sqrt{b\times\overline{b-c}}$; that is,
the diſtance of A from the center of the inſcribed circle is $=\sqrt{b\times\overline{b-c}}$.

PROB. CXI.

Having the ſides of an oblique triangle ABC; *to find the perpendicular* CD, *the ſegments of the baſe, and the area.* 56.

Let AC $=a$, AB$=b$, CB$=c$, $\frac{a+b+c}{2}=z$, $a+c=s$, $a-c=d$.

1. *For the ſegments.*

AB : AC + CB : : AC — CB : dif. ſegments (Geom. II. 24.); that is, $b:s::d:\frac{ds}{b}=$ AD—DB. Then $\frac{1}{2}b+\frac{ds}{2b}$ or $\frac{bb+ds}{2b}=$ greater ſegment AD, and $\frac{bb-ds}{2b}=$ BD the leſſer ſegment.

2. *For the perpendicular.*

$$\text{CD}=\sqrt{aa-\overline{\frac{bb+as}{2b}}\Big|^2}=\sqrt{\frac{4bbaa-b^4-2bbds-ddss}{4bb}}$$

but $2a=s+d$, and $4aa=\overline{s+d}|^2$, and $4bbaa=\overline{bs+bd}$, therefore

CD =

Fig. 56.

$$CD = \sqrt{\frac{bbss + 2bbsd + bbdd - b^4 - 2bbds - ddss}{4bb}} = \sqrt{\frac{ss \times \overline{bb - dd} - bb \times \overline{bb - dd}}{4bb}} = \sqrt{\frac{\overline{bb - dd} \times \overline{ss - bb}}{4bb}}$$

$$= \frac{\sqrt{\overline{b+d} \times \overline{b-d} \times \overline{s+b} \times \overline{s-b}}}{2b}.$$ But $s + b = a + c + b$, $s - b = a + c - b$, and $b + d = a + b - c$, $b - d = b + c - a$, therefore

$$CD = \frac{\sqrt{\overline{a+b+c} \times \overline{a+c-b} \times \overline{a+b-c} \times \overline{b+c-a}}}{2b}.$$

But $z = \frac{a + b + c}{2}$, $z - a = \frac{b + c - a}{2}$, $z - c = \frac{a + b - c}{2}$, $z - b = \frac{a + c - b}{2}$. Therefore

$$CD = \frac{\sqrt{2z \times 2.\overline{z - a} \times 2.\overline{z - c} \times 2.\overline{z - b}}}{2b} = \frac{2\sqrt{z \times \overline{z - a}.\overline{z - b}.\overline{z - c}}}{b},$$ that is, the perpendicular

$$CD = \frac{\sqrt{\overline{a+b+c} \times \overline{a+b-c} \times \overline{a-b+c} \times \overline{b+c-a}}}{2b}$$

$$= \frac{\sqrt{\overline{b+d} \times \overline{b-d} \times \overline{s+b} \times \overline{s-b}}}{2b} =$$

$$\frac{2\sqrt{z \times \overline{z - a} \times \overline{z - b} \times \overline{z - c}}}{b}.$$

3. *For the area.*

Since the area is $= \frac{1}{2}$ AB × CD (Geom. II. 10. cor. 2.), and CD was found by the laſt article, let $CD = p$; ſince $AD = \frac{aa + bb - cc}{2b}$,

therefore CD $= \sqrt{aa - \overline{\frac{aa+bb-cc}{2b}}\Big|^{2}} =$ Fig. 56.
$\sqrt{\frac{2aabb+2aacc+2bbcc-a^4-b^4-c^4}{4bb}}$; therefore we have the area of the triangle ACB $= \frac{1}{2}pb =$
$\frac{1}{2}\sqrt{2aabb+2aacc+2bbcc-a^4-b^4-c^4} =$
$\frac{1}{4}\sqrt{\overline{a+b+c}\times\overline{a+b-c}\times\overline{b+c-a}\times\overline{a+c-b}}$
$= \frac{1}{4}\sqrt{\overline{b+d}\times\overline{b-d}\times\overline{s+b}\times\overline{s-b}} =$
$\sqrt{z\times\overline{z-a}\times\overline{z-b}\times\overline{z-c}}$.

PROB. CXII.

Having the ſides of an oblique triangle; to find the radius of the inſcribed circle, &c. 57.

1. In the triangle ABC, biſect the two angles A, B, by the lines AF, BE to interſect in O the center of the inſcribed circle. From O, C, let fall the perpendiculars OD, CP, upon the baſe AB. And put AB $=b$, AC $=a$, CB $=c$, AP $=d$, PB $=f$, CP $=p$, and DO $=x$, DP $=y$; then AD $=d-y$, BD $=f+y$.

Then (Geom. II. 25.) CA : AP : : CS : SP, and CA + AP : AP : : CP : SP, that is, $a+d : d :: p : \frac{pd}{a+d} =$ SP. Likewiſe $c+f : f :: p : \frac{pf}{c+f}$ $=$ LP. The triangle APS, ADO are ſimilar, and $d : \frac{pd}{a+d} :: d-y : \frac{pd-py}{a+d} =$ DO $= x$. Alſo the triangles BPL and BDO are ſimilar, and $f : \frac{pf}{c+f} :: f+y : \frac{pf+py}{c+f} = x = \frac{pd-py}{a+d}$; then multi-

plying

Fig. plying, $apf+dpf+apy+dpy=cpd+fpd-cpy-fpy$,
57. and tranſpoſing, $apy+dpy+fpy+cpy=cpd-apf$; that is, becauſe $d+f=b$, $apy+bpy+cpy=cpd-apf$, and $y=\frac{cd-af}{a+b+c}$. Whence $x=\frac{pd-py}{a+d}=\frac{pd}{a+d}$

$$-\frac{pcd-pat}{\overline{a+d}\times\overline{a+b+c}}=\frac{pda+pdb+pdc-pdc+paf}{\overline{a+d}\times\overline{a+b+c}}=$$

$$\frac{pba+pbd}{\overline{a+d}\times\overline{a+b+c}}=\frac{pb}{a+b+c}.$$

And ſince p may be had various ways, from the laſt problem; therefore we ſhall have the radius of the inſcribed circle

$$=\frac{bp}{a+b+c}=\frac{1}{2}\sqrt{\frac{\overline{a-c+b}\times\overline{a+c-b}\times\overline{b+c-a}}{a+b+c}}$$

$$=\frac{1}{2}\sqrt{\frac{\overline{b+n}\times\overline{b-n}\times\overline{s-b}}{s+b}}=\sqrt{\frac{\overline{z-a}\times\overline{z-b}\times\overline{z-c}}{z}}$$

Where $z=\frac{a+b+c}{2}$, $s=a+c$, $n=a-c$.

2. *For the tangent* AD.

We have $AD=d-y=d-\frac{cd-af}{a+b+c}=$

$$\frac{ad+bd+cd-cd+af}{a+b+c}=\frac{ad+bd+a\times\overline{b-d}}{a+b+c}=\frac{ba+bd}{a+b+c}$$

But (Geom. II. 23.) $bd=\frac{aa+bb-cc}{2}$, therefore

$$AD=\frac{2ab+aa+bb-cc}{2\times\overline{a+b+c}}=\frac{\overline{a+b}|^2-cc}{2\times\overline{a+b+c}}=$$

$$\frac{\overline{a+b+c}\times\overline{a+b-c}}{2\times\overline{a+b+c}}=\frac{a+b-c}{2};$$ that is, the tangent $AD=\frac{a+d}{a+b+c}b=\frac{a+b-c}{2}=z-c$,

Fig. 57.

3. *For the central diſtance.*

$$AO^2 = AD^2 + DO^2 = \frac{\overline{a+d}|^2}{\overline{a+b+c}^2} bb + \frac{pp}{\overline{a+b+c}^2} bb = \frac{aa+2ad+dd+aa-dd}{\overline{a+b+c}^2} bb = \frac{2aa+2ad}{\overline{a+b+c}|^2} bb = \frac{2abb}{\overline{a+b+c}|^2}$$

$\times \overline{a+d}$. But $d = \dfrac{aa+bb-cc}{2b}$; therefore $AO^2 =$

$$\frac{2abb}{\overline{a+b+c}|^2} \times \frac{2ab+aa+bb-cc}{2b} = ab \times \frac{\overline{a+b}|^2 - cc}{\overline{a+b+c}|^2} = \frac{\overline{a+b+c} \times \overline{a+b-c}}{\overline{a+b+c}|^2} \times ab = \frac{a+b-c}{a+b+c} \times ab = \frac{z-c}{z} ab.$$

That is, the diſtance of the center from the angle A is $= \sqrt{\dfrac{a+b-c}{a+b+c} \times ab} = \sqrt{\dfrac{z-c}{z} \times ab}$.

PROB. CXIII.

Having the ſides of a triangle to find the radius of the circumſcribing circle. 58.

Let ABC be the triangle, draw the diameter CF of the circumſcribing circle, and let CP be perpendicular to AB. Put $AC = a$, $AB = b$, $CB = c$, $CP = p$, $z = \dfrac{a+b+c}{2}$, $s = a+c$, $d = a-c$, CH or HF = R.

Then (Geom. IV. 28), $p : a :: c : 2R$, and $R = \dfrac{ac}{2p}$. Now ſince we have the value of p various ways by problem cxi. we ſhall have the value of R ſo many ways. Hence R the radius of the circumſcribing circle $= \dfrac{ac}{2p}$.

=

Fig. 58.

$$= \frac{abc}{\sqrt{\overline{a+b+c} \times \overline{a+b-c} \times \overline{a-b+c} \times \overline{b+c-a}}}$$

$$= \frac{abc}{\sqrt{\overline{b+d} \times \overline{b-d} \times \overline{s+b} \times \overline{s-b}}}$$

$$= \frac{acb}{4\sqrt{z . \overline{z-a} . \overline{z-b} . \overline{z-c}}}$$

$$= \frac{acb}{4 \times \text{area}}.$$

Cor. *Hence* r *the radius of the inſcribed circle: to* R *the radius of the circumſcribed circle* : : *As* $\overline{z-a} \times \overline{z-b} \times \overline{z-c}$: *to* $\frac{1}{4}abc$.

PROB. CXIV.

59. *Given the baſe of a triangle, and the diameters of the inſcribed and circumſcribed circles; to find the ſides.*

Let QRW be the triangle, QDWB the circumſcribing circle, DB (perpendicular to QW) its diameter. Draw BR, which will biſect the angle R. Let QS biſect the angle Q, then S is the center of the inſcribed circle. Through S draw ASV parallel to QW. Then AP is the radius of the inſcribed circle. Draw BV, BW.

Let BD$=a$, QW$=b$, AP$=c$, BP$=v$, BR$=x$, then $av - vv = \frac{1}{4}bb$, and $v = \frac{a \pm \sqrt{aa-bb}}{2}$. Let BW $= d = \sqrt{vv + \frac{1}{4}bb} = \sqrt{av}$. BA $= v + c = s$, BV $= p = \sqrt{\text{BA} \times \text{BD}}$.

The triangles BRD, BPT and BAS are ſimilar, whence $x : a :: v : \frac{av}{x} =$ BT; and $x : a :: s : \frac{as}{x} =$ BS. But (Geom. IV. 17. cor.) $as = pp$, and $av = dd$; therefore $BT = \frac{dd}{x}$, and $BS = \frac{pp}{x}$. Fig. 95. Then $RS = x - \frac{pp}{x}$, and $TS = \frac{pp - dd}{x}$. But (Geom. II. 25.) TS : SR : : TQ : QR; and the triangles TQR and BWR are ſimilar (Geom. IV. 12. cor. 2.), and TQ : QR : : BW : BR; whence TS : SR : : BW : BR, that is, $\frac{pp-dd}{x} : \frac{xx-pp}{x} :: d : x$, whence $ppx - ddx = dxx - dpp$, and $dxx + ddx = dpp + ppx$, or $\overline{d + x} \times dx = \overline{d + x} \times pp$, and $dx = pp$, whence $x = \frac{pp}{d}$. Then BR (x) : DR ($\sqrt{aa - xx}$) : : BP (v) : $PT = \frac{v\sqrt{aa - xx}}{x}$; whence QT, TW are known. Then BW (d) : BR (x) : : QT : QR : : and TW : WR, the two ſides of the triangle.

PROB. CXV.

There is given the baſe of a triangle, the line that biſects the vertical angle, and the diameter of the circumſcribing circle; to find the ſides. 60.

Let $AB = b$, EO or $OF = r$, $CD = d$, $HD = x$, $FD = y$.

Then $AD = \frac{1}{2} b + x$, $DB = \frac{1}{2} b - x$, $FH = \sqrt{yy - xx}$.

And (Geom. IV. 20. cor. 2.) ADB = CDF, or $\frac{1}{4} bb - xx = dy$. The triangles FDH, FEC are ſimilar, and $y : \sqrt{yy - xx} :: 2r : y + d$, and $yy + dy = 2r\sqrt{yy - xx} = 2r\sqrt{yy + dy - \frac{1}{4}bb}$. Which ſquared is, Fig. 60. $y^4 + 2dy^3 + ddyy = 4rryy + 4rrdy - rrbb$, and reduced $y^4 + 2dy^3 \begin{array}{l} + ddyy \\ - 4rr \end{array} - 4rrdy + rrbb = 0$. Then $x = \sqrt{\tfrac{1}{4}bb - dy}$.

Alſo BF $= \sqrt{yy - xx + \tfrac{1}{4}bb}$, and the triangles ADF, CDB are ſimilar, and AD $(\tfrac{1}{2}b + x)$: AF or BF $(\sqrt{yy - xx + \tfrac{1}{4}bb})$: : CD (d) : CB $= \dfrac{d\sqrt{yy - xx + \tfrac{1}{4}bb}}{\tfrac{1}{2}b + x}$.

Alſo the triangles ADC, BDF are ſimilar, and BD $(\tfrac{1}{2}b - x)$: BF $(\sqrt{yy - xx + \tfrac{1}{4}bb})$: : CD (d) : CA $= \dfrac{d\sqrt{yy - xx + \tfrac{1}{4}bb}}{\tfrac{1}{2}b - x}$.

* * *

SECT.

SECT. VII.

Problems in Plain Trigonometry.

PROB. CXVI.

In the triangle ABC, *there is given the angle* B, *the* 61.
ſide AB; *and the ſum of the ſides* BC, AC; *to find the ſides.*

LET $AB = d$, $BC + AC = b$, ſine $< B = s$, coſ. $= c$, $AC = x$, then $CB = b - x$.

By plain Trigonometry rad. (1) : AB (d) : : S.PAB or coſ. B (c) : $PB = cd$.

Then (Geo. II. 22.) $xx = dd + bb - 2bx + xx + 2cd \times \overline{b - x}$, reduced $2bx + 2cdx = bb + dd + 2bcd$, and $x = \frac{bb + dd + 2bcd}{2b + 2cd}$.

PROB. CXVII.

In the triangle ACB, *there is given the two ſegments* 62.
AD, DB, *made by the perpendicular, and the angle* ACB; *to find the reſt.*

Make $DE = DB$, and draw CE, then put $BD = b$, $AD = d$, CB or $CE = y$, $S.ACB = s$, $S.ACE = x$; then $AE = d - b$, $AB = d + b$.

By Trigonometry, (in the triangle ACB) AB ($b + d$) : S.ACB (s) : : CB (y) : $\frac{sy}{b + d} = S.CAB$.

Alſo (in the triangle ACE), CE (y) : S.CAE $\left(\frac{sy}{b+d}\right)$: : AE ($d - b$) : S.ACE (x), and $x = \frac{d-b}{d+b}s$.

Fig. 62. Then $\frac{ACE + ACB}{2} = ACD$, and $\frac{ACB - ACE}{2} = BCD$. Then S.ACD : AD : : rad : AC. And S.BCD : BD : : rad : CB.

PROB. CXVIII.

63. *In the triangle* ABC *there is given* AB, *and the angle* B, *and the ratio of* AC *to* BC, *to find the ſides.*

Let fall AD on BC (produced); and put AB $= b$, AC $= a$, the ratio of AC to CB as 1 to r, then CB $= ra$, and coſ. ACB $= c$. Then rad. (1) : AC (a) : : S.DAC (c) : $ca =$ DC. Then (Geom. II. 22.) $bb = aa + rraa + 2craa$, whence $aa = \frac{bb}{rr + 1 + 2cr}$, and $a = \frac{b}{\sqrt{rr + 1 + 2cr}}$.

PROB. CXIX.

64. *In the triangle* CAB, *there is given two ſides and the included angle; to find the area.*

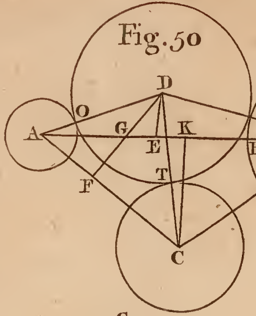
Let CA, AB and the angle A be given; draw CF perpendicular to AB, and let AB$=b$, AC$=d$, S. $<$ A $= s$. Then in the triangle ACF, rad. (1) : AC (d) : : S.A (s) : $sd =$ CF. Then $\frac{CF \times AB}{2}$ $=$ area, or $\frac{sdb}{2} =$ area; that is, half the rectangle of the ſides multiplied by the ſine of the included angle, gives the area.

PROB. CXX.

65. *Given all the ſides of a trapezium, and two oppoſite angles; to find the area.*

Let the angles B, D be given, and through the other two angles A, C, draw the diagonal AC. Let

Fig.50



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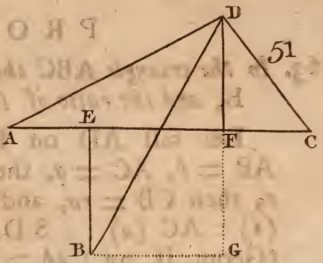

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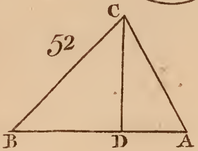

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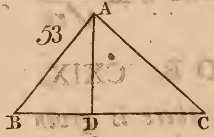

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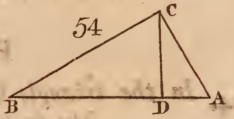

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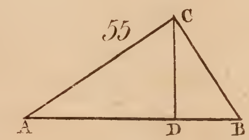

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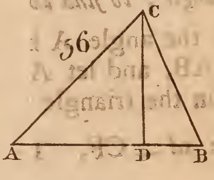

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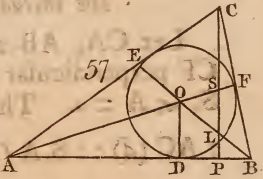

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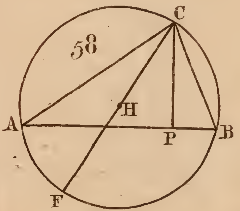

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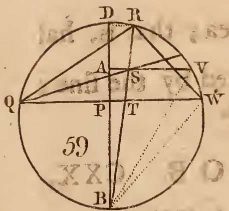

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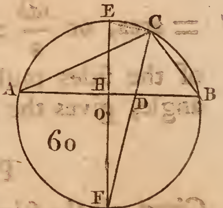

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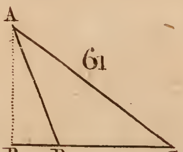

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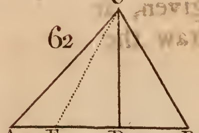

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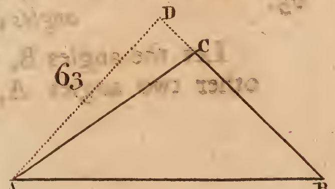


Pl.VI. *pa.420*

Let AB $=b$, BC $=c$, CD $=d$, DA $=f$, S. $<$B $=p$, Fig.
S.D $=q$. Then by the laſt problem, the area of 65.
the triangle CBA $=\frac{bcp}{2}$, and the triangle CDA $=\frac{dfq}{2}$; therefore the trapezium $=\frac{bcp+dfq}{2}$.

PROB. CXXI.

In the triangle WNE, *there is given the ſegment* SE, 66.
the angle WNE, *and the ratio of* NE *to* NW; *to find the ſides.*

Let WP be perpendicular to EN; and ſuppoſe NE to WN as 1 to p, S.N $=s$, coſ. N $=c$, SE $=b$, NE $=x$, then WN $=px$. In the triangle WNP, rad. (1) : WN (px) : : S.PWN (c) : PN $=pcx$, and by the ſimilar triangles ENS, EWP; ES : EN : : EP : EW, or $b : x : : x+pcx \quad \frac{xx+pcxx}{b}$ = EW. But (Geom. II. 22.)

$$\frac{x^4+2pcx^4+ppccx^4}{bb}=xx+ppxx+2pcxx,$$

or

$$\overline{1+2pc+ppcc}\times xx=bb\times\overline{1+pp+2pc},$$

and

$$x=\frac{b}{1+p}\sqrt{1+pp+2pc}.$$

Or thus,

Let $t=$ tang. of $\frac{\text{EWN}+\text{E}}{2}$, then WN+NE ($px+x$) : WN—NE ($px-x$) : : $t : \frac{tpx-tx}{px+x}=\frac{p-1}{p+1}t=$ tang. $\frac{1}{2}$ diff. of the angles W and E. Whence the angles W, E are known. Then as coſ. E : b : : rad : x, required.

PROB. CXXII.

67. *In the right-angled triangle* ABC, *there is given, the ſum of* AB *and* BC, *the angle* CDB; *likewiſe* $\angle$ACD $=\angle$DCE *are given; to find* CB, *&c.*

Let S.ACB $=s$, S.CAB $=c$, CB $=x$, AB $+$ BC $=b$, and BA $=b-x$. Then $s:c::b-x:x$, and $sx=cb-cx$, whence $x=\frac{cb}{s+c}$. Then CB, BA are known. Let m, n be the tangents of BCE, BCD; then $1:x::m:$ BE $::n:$ BD.

PROB. CXXIII.

In the triangle ADC, *there is given* AB, BC; *and the angles* ADB, BDC; *to find* AD, DC.

Let S.ADB $=s$, S.BDC $=t$, S.ADC $=p$, cotang. ADC $=q$, AB $=b$, BC $=c$, AC $=d$, AD $=x$.

By plain Trigonometry, $b:s::x:\frac{sx}{b}=$ S.ABD or CBD. And $t:c::\frac{sx}{b}:\frac{csx}{tb}=$ CD. Then AD $+$ CD $\left(x+\frac{csx}{tb}\right)$: AD $-$ CD $\left(x-\frac{csx}{tb}\right)$ :: tan. $\frac{A+C}{2}$ (q) : $\frac{tbq-csq}{tb+cs}=$ tang. $\frac{A-C}{2}$. Then $\angle$s A and C are known; then $p:d::$ S.C : AD :: S.A : CD.

PROB. CXXIV.

69. *In the triangle* ABC, *there is given* AB, *the angle* C, *and* CD, *which is drawn to the middle of* AB; *to find the ſides.*

Draw AF perpendicular to CB, and put AD or DB $=b$, CD $=d$, S.$\angle$C $=s$, coſ. C $=c$, AC $=x$, BC $=y$.

Then (Geom. II. 28.) $xx+yy=2bb+2dd$. By Trigonometry $1:x::s:sx=$ AF, and $1:x::c:cx=$ CF. Then $\text{BF}=y-cx$, and $ssxx+yy-2cyx+ccxx=4bb$, subtract this from the first equation, then $xx-ssxx+2cyx-ccxx=2dd-2bb$, that is (because $ss+cc=1$.) $2cyx=2dd-2bb$, and $2yx=\frac{2dd-2bb}{c}$; therefore $yy+2yx+xx=2bb+2dd+\frac{2dd-2bb}{c}$, and $y+x=\sqrt{2bb+2dd+\frac{2ad-2bb}{c}}=m$; in like manner $y-x=\sqrt{2bb+2dd-\frac{2dd-2bb}{c}}=n$. Then $y=\frac{m+n}{2}$, $x=\frac{m-n}{2}$. Fig. 69.

PROB. CXXV.

Given the angles of altitude BCA, BDA, *the horizontal angle* BCD, *and the line of station* CD; *to find the height* AB. 70.

Let $b=$ cotang. BCA $=47:30$; $c=$ cotang. BDA $=40:12$; $d=$ S.BCD $=87\ 5$; CD $=f=283.274$ feet; AB $=x$.

Then in the triangle ABC, $1:x::b:bx=$ BC, and in the triangle ABD, $1:x::c:cx=$ BD; and in the triangle BCD, $cx:d::bx:\frac{db}{c}=$ S.BDC $=50\ 39$; whence DBC $=42\ 16$; let $n=$ S.DBC; then $n:f::d:\text{BD}=cx$, and $ncx=fd$, and $x=\frac{fd}{nc}=355.458$.

Fig.

PROB. CXXVI.

71. *Given the ſum of the ſides of a triangle, and all the angles ſeverally; to find the ſides.*

Let S.A $=s$, S B $=n$, S.C $=t$, AB+BC+CA $=b$, AC $=x$. Then by Trigonometry, $n : x :: s : \frac{sx}{n}$ = CB, and $n : x :: t : \frac{tx}{n}$ = AB. And $x + \frac{sx}{n} + \frac{tx}{n} = b$, whence $x = \frac{nb}{n+s+t}$.

PROB. CXXVII.

72. *In the right-angled triangle* VAB, *there is given the perpendicular* AB, *the ſegment* VC, *and the angle* VAC; *to find* CB.

Let AB $= b$, VC $= c$, tang. VAC $= t$, BC $= a$. Then by plain trigonometry, $b : 1 :: a : \frac{a}{b}$ = tang. BAC, and (trig. viii.) $1 - \frac{ta}{b} : 1 :: t + \frac{a}{b} : \frac{bt+a}{b-ta}$ = tang. BAV. Whence $1 : b :: \frac{bt+a}{b-ta} : a + c = \frac{bbt+ba}{b-ta}$, and multiplying, $ba + bc - taa - tca = bbt + ba$, reduced $aa + ca = \frac{bc}{t} - bb$.

PROB. CXXVIII.

73. *In the right-angled triangle* ABC, BE = EC, *and* < ABD = CBD; *and there is given* BD *and* < CAE; *to find the ſides.*

Draw DF parallel to CB; then in the triangle DFB, the angles at B, D are 45°, and BD being given, DF and FB its equal, are given; and ſince Fig. CE = EG, therefore DG = GF. Let DG or GF $= b$, 73. S.DAG $= s$, AF $= x$, then AG $= \sqrt{bb+xx}$, AD $= \sqrt{4bb+xx}$. And by plain Trig. AG$(\sqrt{bb+xx})$: rad. (1) : : AF (x) : $\frac{x}{\sqrt{bb+xx}}$ = S.G. Alſo S. G $\left(\frac{x}{\sqrt{bb+xx}}\right)$: AD $(\sqrt{4bb+xx})$: : S.DAG (s) : DG (b); whence $\frac{bx}{\sqrt{bb+xx}} = s\sqrt{4bb+xx}$, reduced

$$x^4 + \begin{matrix} 5bbxx \\ -\frac{bbxx}{ss} \end{matrix} + 4b^4 = 0.$$

PROB. CXXIX.

From the point B, *to draw the lines* BC, BD, BA, 74.
ſo that CD, DA, *and the angles* CBD, DBA,
may be given.

Draw CF perpendicular to AB, and put CD $= b$, DA $= c$, CA $= s$, and S.CBD $= f$, S.DBA $= d$, S.CBA $= m$. coſ. CBA $= n$, and CB $= x$.

Then by plain Trigonometry $1 : x : : n : nx =$ BF, and $b : f : : x : \frac{fx}{b}$ = S.D, and $d : c : : \frac{fx}{b} : \frac{cfx}{bd}$ = BA. But in the triangle CBA, (Geom. II. 23.) $ss = xx + \frac{ccffxx}{bbdd} - 2nx \times \frac{cfx}{bd}$, which reduced is

$$x = \frac{bds}{\sqrt{bbdd + ccff - 2bdfnc}}.$$

PROB.

PROB. CXXX.

75. *In an oblique triangle there is given, the baſe, and perpendicular, and angle oppoſite to the baſe; to find the ſides.*

Draw AD perpendicular to CB; and put AB $= b$, S.ACB $= s$, coſ. ACB $= c$, perp. CF $= p$, AC $= x$, CB $= y$.

In the triangle ACD, $1 : x :: s : sx =$ AD, and $1 : x :: c : cx =$ CD. The triangles ABD and CBF are ſimilar, and $y : b :: p : sx =$ AD, whence $pb = sxy$. In the triangle ABC (Geom. II. 23.), $bb = xx + yy - 2cxy$; but $xy = \frac{pb}{s}$, and $y = \frac{pb}{sx}$; therefore $bb = xx + \frac{ppbb}{ss} - \frac{2cpb}{s}$: which reduced is $x^4 \begin{array}{l} -bb \\ -\frac{2bcp}{s} \end{array} xx + \frac{bbpp}{ss} = 0$.

Otherwiſe,

Let AC+CB $= x$, AC—CB $= y$, the reſt as before; then AD $= s \times \overline{x+y}$, CD $= c \times \overline{x+y}$, $\frac{pb}{s} = xx - yy$. Then in the triangle ABC, $bb = \overline{x+y}|^2 + \overline{x-y}|^2 - 2c \times \overline{xx - yy}$; that is, $2xx + 2yy - 2cxx + 2cyy = bb$, and putting $xx - \frac{pb}{s}$ for yy, we have $2xx + 2xx - \frac{2pb}{s} - 2cxx + 2cxx - \frac{2cpb}{s} = bb$, or $4xx - \frac{2pb}{s} - \frac{2cpb}{s} = bb$, whence $x = \sqrt{\frac{1}{4}bb + \frac{1+c}{2s} \times bp} = m$, and $y = \sqrt{\frac{1}{4}bb + \frac{c-1}{2s}bp} = n$. Then AC $= \frac{m+n}{2}$ BC $= \frac{m-n}{2}$.

Or thus, Fig.

Let $f =$ coſine of the ſum of the angles A, B; 75.
$v =$ coſ. of their difference; the reſt as before. Then (Trig. II. 10.) $v - f : s :: p : \frac{1}{2}b$, and $v - f = \frac{2ps}{b}$, and $v = \frac{2ps}{b} + f$. Then the angles A and B will be known, and conſequently their oppoſite ſides.

PROB. CXXXI.

Given all the ſides of a triangle, to find the center of the circumſcribed circle. 76.

On the middle of AB, AC, erect the perpendiculars DO, FO; the point of interſection O, is the center of the circumſcribing circle. From O draw OI, OG, parallel to AC, AB; and put AB $= b$, AC $= d$, S.$\angle$A $= s$, coſ. A $= c$, AI $= x$, IO $= y$. The $\angle$CGO $=$ CAB $=$ OID; and in the right-angled triangles OID, OGF, it will be $1 : y :: c : cy =$ ID, and $1 : y :: s : sy =$ OD. Alſo $1 : x :: c : cx =$ GF, and $1 : x :: s : sx =$ FO. Then $x = \frac{1}{2}b - cy$, and $y = \frac{1}{2}d - cx$, and $cy = \frac{1}{2}b - x = \frac{1}{2}cd - ccx$, and $x - ccx = \frac{1}{2}b - \frac{1}{2}cd$, or $ssx = \frac{b-cd}{2}$, whence $x = \frac{b-cd}{2ss}$, and $y = \frac{d-cb}{2ss}$. Therefore DO $= \frac{d-cb}{2s}$, and FO $= \frac{b-cd}{2s}$. Likewiſe AO $= \frac{\sqrt{bb+dd-2bcd}}{2s}$.

PROB. CXXXII.

In the given triangle ABC, *the angles* AOC, COB, BOA, *about the point* O, *are given; to find the diſtances* AO, BO, CO. 77.

Produce CO to D, and BO to E. And let AB $= b$, AC $= d$, CB $= f$, S.A $= s$, coſ. A $= c$, S.B $= q$,

Fig. 77. S.B $= q$, coſ. B $= n$, S.AOE $= g$, coſ. AOE $= m$, S.AOD $= b$, S DOB $= p$, AO $= x$.

Then by Trigonometry, AB (b) : S.O (g) : : AO (x) : $\frac{gx}{b}$ = S.ABO, and $\sqrt{1 - \frac{ggxx}{bb}}$ = coſ. ABO $= y$; and (Trig. I. 6. cor.) $\frac{ggx}{b} + my$ = coſ. OAB; and (I. 6.) $gy - \frac{mgx}{b}$ = S.OAB. Alſo (Trig. I. 6.) $\frac{sggx}{b} + smy - cgy + \frac{cmgx}{b}$ = S.CAO; and (I. 6.) $qy - \frac{gnx}{b}$ = S.CBO. And by Trigonometry, $p : f : : qy - \frac{gnx}{b} : \frac{fqy}{p} - \frac{gnfx}{bp}$ = CO, and $b : d : :$ S.CAO : CO $= \frac{sggdx}{bb} + \frac{dsmy}{b} - \frac{dcgy}{b} + \frac{dcmgx}{bb} = \frac{fqy}{p} - \frac{gnfx}{bp}$, and multiplying, $sggdpx + bpdsmy - bpdcgy + pdcmgx = bbfqy - gnfbx$; and tranſpoſing, $bbfqy + bpdcgy - bpdsmy = sggdx + pdcmgx + gnfbx$, and $y = \frac{sggd + pdcmg + gnfb}{bbfq + bpdcg - bpdsm} x = \frac{rx}{t}$ by ſubſtitution, that is, $\sqrt{1 - \frac{ggxx}{bb}} = \frac{rx}{t}$, and $1 - \frac{ggxx}{bb} = \frac{rrxx}{tt}$, and $bbtt - ggttxx = bbxx$, reduced

$$x = \frac{bt}{\sqrt{bb + ggtt}}.$$

Or thus,

78. Make the angle BAF = ſupplement of BOC, and <ABF = ſup. AOC; through A, B, F deſcribe the circle AOBF, to interſect CF in O, the point required.

Calculation. In the triangle ABF, all the angles are given, and the ſide AB, to find AF. Fig. 78.

In the triangle CAF; CA, AF, and <CAF are given; to find <ACF.

In the triangle ACO, there is given AC, and all the angles; to find AO, CO.

PROB. CXXXIII.

In the right-angled triangle ABC, *there is given* BA, 79. *and angle* CBD; *alſo* AT=TD, *and* <ABT=CBT; *to find* AC.

Let BA $= b = 95.23$, tang. DBC$= t =$T. 15°, AT or DT$= a$, then AD$= 2a$. By trigonometry, $b : a :: 1 : \frac{a}{b} =$tang. TBA, and $b : 2a :: 1 : \frac{2a}{b} =$ tang. DBA. Then (Trig. I. 2. Schol.) the tang. 2TBA or tang. ABC $= \frac{2ab}{bb-aa}$. Alſo (Trig. I. 8.) $1 - \frac{2ta}{b} : 1 :: t + \frac{2a}{b} : \frac{bt+2a}{b-2ta} =$ tang. $\overline{\text{ABD} + \text{DBC}} =$ tang. ABC. Whence $\frac{2ab}{bb-aa} = \frac{bt+2a}{b-2ta}$, reduced $2a^3 - 3btaa = b^3t$, and $a = 65.13$, and $\frac{2ab}{bb-aa} \times b =$ AC.

PROB. CXXXIV.

Upon a horizontal plane, there ſtands a tall pine-tree leaning towards the ſouth. A man ſtanding on the north ſide of it 50 yards from the foot, finds the tree to ſubtend an angle of 39 deg. Afterwards going directly weſt 73 yards, it ſubtends an angle of 46°. What is the tree's length?

Let AC be the tree; E, F, the two ſtations. 80. Draw AB perpendicular to the horizon, and AD perpen-

Fig. 80. perpendicular to FC produced; draw BD, AC. The triangles ABD, ABC, ADC, BDC, CEF, DAF, BAE, are all right-angled. Put EF$=b$, CE$=d$, CF$=c$, CD$=y$, tang. AEB$=s$, tang. AFC$=t$.

By Trigonometry, $1 : t :: c+y : \overline{c+y} \times t =$ AD, whence AC $= \sqrt{yy + tt \times \overline{c+y}}$. The triangles FCE, BCD are ſimilar, and $d : c :: y : \frac{cy}{d} =$ BC, and $d : y :: b : \frac{by}{d} =$ BD. Then AB $= \sqrt{yy + tt \times \overline{c+y}|^2 - \frac{ccyy}{dd}} = \sqrt{tt \times \overline{c+y}|^2 - \frac{bbyy}{dd}}$. And in the triangle ABE, $d + \frac{cy}{d} : 1 :: \sqrt{tt \times \overline{c+y}|^2 - \frac{bbyy}{dd}} : s$, whence $\frac{dds + csy}{d} = \sqrt{tt \times \overline{c+y}|^2 - \frac{bbyy}{dd}}$; and ſquared $d^4ss + 2cddssy + ccssyy = dd \times \overline{ttcc + 2ttcy + ttyy} - bbyy$, and reduced

$$\begin{array}{lll} ccss\,yy & +\ 2cddss\,y & =\ ddttcc. \\ -ddtt & -\ 2cddtt & \quad -\ d^4ss \\ +\ bb & & \end{array}$$

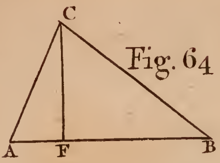
PROB. CXXXV.

81. *Given two altitudes and two azimuths of a cloud in motion; to find the point of the wind.*

Let A be the firſt, B the ſecond place of the cloud, O the place of obſervation, ABC the plane of the cloud's motion; AB its line of direction. Let AD, BE, CO be perpendicular to the horizon, then DEO is equal and parallel to ABC, and MDE is the path of the cloud on the earth. Let OM be the meridian.

Put $p=$tang. AOD, $q=$tang. BOE, $t=$cotang. $\frac{1}{2}$DOE, OD $= x$. In the triangle AOD, $1 : x :: p : px =$ AD $=$ BE; and in the triangle BOE, q :

Fig. 64



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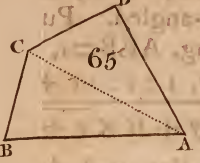

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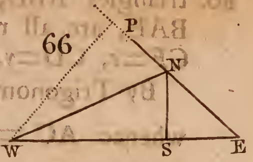

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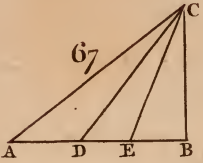

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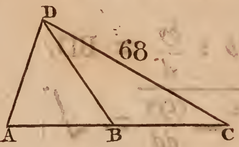

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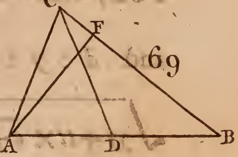

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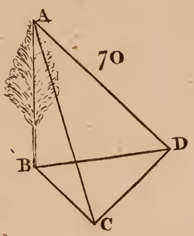

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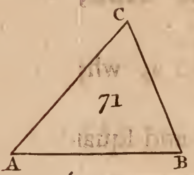

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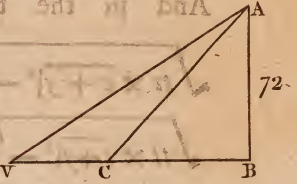

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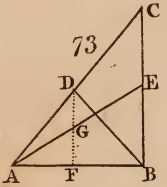

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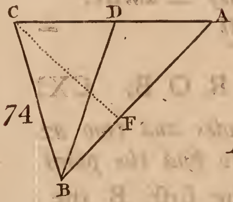

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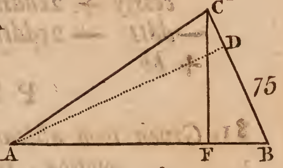

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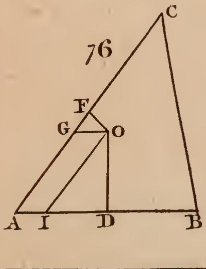

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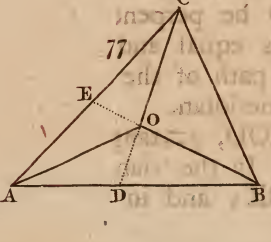

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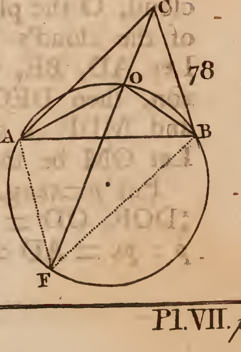


Pl.VII. *pa.* 430

$q : px :: 1 : \frac{px}{q} = OE$. In the triangle DOE, (Trig. Fig. 81. II. 6.) $\frac{px}{q} + x : \frac{px}{q} - x :: t : \frac{p-q}{p+q}t =$ tang. $\frac{ODE-OED}{2}$, and the ſum and difference of ODE, OED being had; ODE, and OED will be known.

In the triangle ODM, there is given ODM and DOM, therefore OMD is known, which is the way of the cloud or of the wind.

PROB. CXXXVI.

On a clear day, the wind ſtanding N. N. E. *I obſerved a ſmall cloud* W. *by* S. *whoſe altitude was* 41°, *and whilſt the ſhadow of the cloud moved over* 1230 *yards upon a horizontal plane, the cloud itſelf moved through an angle of* 9° : 37′ *as I obſerved it with an inſtrument. What was the cloud's height?*

Let E be the place of obſervation, CA the tract 82. of the cloud, FG its projection upon the horizon; AF, CG, BE being perp. to the horizon. Let BD be perp. to ACD, and AK to ECK.

Let AC or GF$=d=1230$, S.DCB$=c=5$ points, coſ. DCB$=n$, S.GEC$=b=41°$, coſ. GEC$=s$, tang. AEC$=t=9° : 37$; CG$=x$. By Trigonometry, in the triangle CGE, $b : x :: 1 : \frac{x}{b} =$ CE, and $b : x :: s : \frac{sx}{b} =$CB, and in the triangle BCD, $1 : \frac{sx}{b} :: c : \frac{csx}{b} =$DB, and $1 : \frac{sx}{b} :: n : \frac{nsx}{b} =$CD. Then DE $= \sqrt{xx + \frac{ccssxx}{bb}} = \frac{x}{b}\sqrt{bb+ccss} = \frac{x}{b}p$, by ſubſtitution.

The

Fig. 82. The triangles CED and CAK are ſimilar, and $\frac{x}{b}$ (CE) : $\frac{px}{b}$ (DE) : : (AC) $d : pd =$ AK; and $\frac{x}{b}$ (CE) : $\frac{snx}{b}$ (CD) : : d (AC) : $snd =$ CK. And in the triangle AKE, $snd + \frac{x}{b}$ (EK) : 1 (rad.) : : pd (AK) : t (tang. AEK), whence $sndt + \frac{tx}{b} = pd$, and $tx = bpd - sndtb$, and $x = \frac{bpd - sndtb}{t} = \frac{bdp}{t} - sndb$.

PROB. CXXXVII.

83. *In the triangle* ACB, *there is given* AC, CB *the ſegment* BD, *and the angle* ACB; *to find the reſt.*

Draw AF perpendicular to CB; and put AC $= a$, DB $= b$, CB $= d$, S.ACD $= s$, coſ. ACD $= c$, and S.CDB $= x$.

In the triangle CDB, $d : x : : b : \frac{bx}{d} =$ S.DCB, and in the triangle CAD, $x : a : : s : \frac{as}{x} =$ AD, then AB $= b + \frac{as}{x} = \frac{bx + as}{x}$. But (Trig. I. 5. cor. 1.) $c\sqrt{1 - \frac{bbxx}{dd}} - \frac{sbx}{d} =$ coſ. ACB $= cz - \frac{sbx}{d}$ (putting $z = \sqrt{1 - \frac{bbxx}{dd}}$); and in the triangle ACF, $1 : a : : cz - \frac{bsx}{d} : cza - \frac{absx}{d} =$ CF. But (Geom. II. 23.) $\frac{\overline{bx + as}|^2}{xx} = aa + dd - 2d \times \overline{caz - \frac{absx}{d}}$, and multiplying by xx, $bbxx + 2absx + aass = aaxx + ddxx - 2dcaxxz + 2absx^3$, and tranſpoſing Fig. 83.

$$\begin{array}{l} 2dcaxxz = 2absx^3 + aaxx - 2absx - aass, \\ + dd \\ - bb \end{array}$$

and reſtoring z, and ſubſtituting for the known quantities, $2dacxx\sqrt{1 - \frac{bbxx}{dd}} = px^3 + qx^2 - rx - t$, and ſquaring, $4ddaaccx^4 - 4aabbccx^6 =$

$$\begin{array}{llllll} p^2x^6 + 2pqx^5 & - 2prx^4 & - 2ptx^3 & - 2qtx^2 & & \\ & + qq & - 2qr & + rr & & \\ + 2rtx + tt. & & & & & \end{array}$$

* * *

SECT. VIII.

Problems in Spherical Trigonometry.

PROB. CXXXVIII.

Given the latitude of the place and the ſun's longitude; to find the aſcenſional difference.

34. LET $b =$ S. greateſt declination, $c =$ S. ſun's longitude AD, $x =$ S. declination BD, $t =$ tang. latitude PCH.

Then in the r^t $<$ ſpherical triangle ADB, rad. (1) : S.AD (c) : : S.A (b) : $bc =$ S.BD, and (Trig. I. 1. ſchol.) tang. BD $= \frac{bc}{\sqrt{1-bbcc}}$; and in the triangle CBD, rad. (1) : co-tang. C (t) : : tang. BD $\left(\frac{bc}{\sqrt{1-bbcc}}\right)$: S.CB (x), and $x = \frac{bct}{\sqrt{1-bbcc}}$.

PROB. CXXXIX.

34. *Given the ſun's declination, and the ſum of the latitude and amplitude; to find each of them.*

Let $a =$ ſine of half the ſum of CD and PH, $b =$ the coſine, $d =$ S. declination, $s =$ S. ſum of CD and PH, $x =$ S. half their difference, $y =$ S. whole difference. Suppoſe PH greater than CD. Then (Trig. I. 6. ſchol.) $b\sqrt{1-xx} - ax =$ coſ. PH; and $a\sqrt{1-xx} - bx =$ S.CD. But in the triangle CBD, coſ. C ($b\sqrt{1-xx} - ax$) : S.BD (d) : : rad. (1) :

S.CD $(a\sqrt{1-xx} - bx$. Whence $ab \times \overline{1-xx} -$ Fig. 84.
$aax\sqrt{1-xx} - bbx\sqrt{1-xx} + abxx = d$;
or $ab - x\sqrt{1-xx} = d$ (becauſe $aa + bb = 1$), But (Trig. l. 2. ſchol.) $2x\sqrt{1-xx} = y$, and $2ab = s$. Whence $s - y = 2d$, and $y = s - 2d$.

PROB. CXL.

Given the ſun's altitude at ſix, and alſo when weſt; to find the latitude. 85.

R is the ſun's place when weſt, and O at ſix a clock. Let $t =$ S.RC the altitude weſt, $s =$ S.OI the altitude at ſix, $x =$ S. latitude.

In the triangle CIO, S.C (x) : S.OI (s) : : rad. (1) : $\frac{s}{x} =$ S.CO the declination. In the triangle DCR, S.C (x) : S.DR $\left(\frac{s}{x}\right)$: : rad. (1) : S.CR (t), and $tx = \frac{s}{x}$, or $xx = \frac{s}{t}$, and $x = \sqrt{\frac{s}{t}}$.

PROB. CXLI.

AEC, BCD *are two triangles right-angled at* E *and* D, *and ſtanding on the great circle* ECD; *alſo* AC $=$ CB, *and* EC, CD, *and the angle* ACB *are given; to find the angles and ſides.* 86.

Let $a =$ tang. DC, $b =$ tang. CE, $s =$ S. half the ſum of the angles BCD, ACE; $c =$ coſine, $x =$ ſine, $y =$ coſine of half the difference. Then $sx + cy =$ coſ. leſſer ACE, and $cy - sx =$ coſ. greater BCD. And in the triangle ACE, $cy + sx : 1 : : b : \frac{b}{cy + sx} =$ tang. AC. And in the triangle BCD,

$cy -$

Fig. 86. $cy - sx : 1 :: a : \frac{a}{cy-sx} =$ tang. CB. Whence $\frac{a}{cy-sx} = \frac{b}{cy+sx}$, and $acy + asx = bcy - bsx$, and $asx + bsx = bcy - acy$, and $\frac{x}{y} = \frac{bc-ac}{bs+as} =$ tang. half the difference of the angles BCD, ACE; whence the angles themſelves are had.

PROB. CXLII.

87. *Given the ſun's amplitude, and altitude at ſix; to find the latitude, and declination.*

Let P be the pole, Z the zenith, CB the amplitude, AP the altitude at ſix.

Let S.CB$=b$, $p=$S.AP, $x=$S. lat. PO, $y=$S. twice the latitude. In the triangle CBD, rad. (1) : S.C $(\sqrt{1-xx})$ :: S.CB (b) : $b\sqrt{1-xx} =$ S.BD or AC. And in the triangle CAP, rad. (1) : S.AC $(b\sqrt{1-xx})$ :: S.C (x) : S.AP (p); therefore $bx\sqrt{1-xx} = p$, and $2x\sqrt{1-xx} = \frac{2p}{b}$, but $y = 2x\sqrt{1-xx}$; whence $y = \frac{2p}{b}$; then x will be known, and $b\sqrt{1-xx}$, the declination.

PROB. CXLIII.

88. *Given two altitudes and two azimuths of the ſun; to find the latitude.*

Let Z be the zenith, P the pole; S, O two places of the ſun. Let s, $ſ$=ſine and coſine of ZS; p, q=ſine and coſine of ZO; m=coſ. PZS, n=coſ. PZO; x, y=ſine and coſine of PH. Then (Trig. II. 38.) coſ. SP $= sym + ſx$, and coſ. OP =

OP $= pyn + qx$. Whence $sym + fx = pyn + qx$, and Fig. 88.
$fx - qx = pyn - sym$; therefore $\frac{x}{y} = \frac{pn-sm}{f-q} =$ tang. PH the latitude.

PROB. CXLIV.

Given the latitude of the place; and the ſun's altitude is equal to his azimuth from the ſouth, and equal to the hour from noon; to find any of them. 89.

Z is the zenith, P the pole, ZP is given, and $\angle ZPO = AZO = DO$. Let $s =$ S.ZP, $c =$ coſ. ZP, $y =$ S.ZPO $=$ S.AZO. Then $\sqrt{1-yy} =$ S.ZO, $y : \sqrt{1-yy} :: y : \sqrt{1-yy} =$ S.OP, and $y =$ coſ. OP. But (Trig. III. 38.) $s\sqrt{1-yy} \times \sqrt{1-yy} + cy = y$, or $s - syy + cy = y$, and $syy + \begin{matrix} y \\ -cy \end{matrix} = s$, or

$$yy + \frac{1-c}{s}y = 1.$$

PROB. CXLV.

There are two places, whoſe latitudes are the complements of each other to 90°, and the ſun's declination being given, he riſes an hour ſooner in one place than the other; to find the latitudes.

Let $t =$ tang. declination, $b =$ tang. aſcenſional difference, $x =$ tang one latitude, then $\frac{1}{x} =$ tang. the other latitude. In the triangle CDB, rad (1) : cotang. DCB $(x) :: $ T.DB $(t) : tx =$ S.DC the aſcenſional difference in the firſt latitude, and $1 : \frac{1}{x} :: t : \frac{t}{x} =$ the aſcenſional difference in the other latitude. But (Trig. I. 9) $1 + tt : 1 : :$ 87.

Fig. 87. $tx - \frac{t}{x} : b$, and $tx - \frac{t}{x} = b + btt$, $txx - t = bx + bttx$, and $xx \begin{matrix} -btx \\ -\frac{b}{t} \end{matrix} = 1$.

PROB. CXLVI.

The ſtile of an horizontal dial being turned down, fell upon the hour line of 8; *query, the latitude it was made for.*

Let t = tang. of 4 hours or 60°, anſwering to 8 o'clock, x = S. latitude; then $\frac{x}{\sqrt{1-xx}}$ = tang. latitude = hour angle of 8, by the queſtion. Whence by the known proportion of dialling, $1 : x :: t : \frac{x}{\sqrt{1-xx}}$, and $tx = \frac{x}{\sqrt{1-xx}}$, or $t\sqrt{1-xx} = 1$, and $\sqrt{1-xx} = \frac{1}{t}$ = coſ. lat.

PROB. CXLVII.

To find in what latitude, an erect ſouth declining dial may be made, ſo that the declination of the plane, the diſtance of the ſubſtile from the meridian, and the ſtile's height, are all equal.

90. Let ABC be the right-angled ſpherical triangle, in which are found all the requiſites; *viz* AB = co-lat. ∠A = co-declination, ∠B = plane's dif. longitude, CB = ſtile's height, AC = ſubſtile's diſtance from the meridian.

Let x = S.AB, y = S.BC. Then (by the properties of right triangles) 1 : tang. BC :: cotang. A : S.AC, or S.AC = tang. BC × cotang. A. But by the queſtion, BC=AC=comp. A. Therefore Fig. 90. tang. BC or cotang. A $= \frac{y}{\sqrt{1-yy}}$. Whence y (S.AC) $= \frac{y}{\sqrt{1-yy}} \times \frac{y}{\sqrt{1-yy}}$ (tang. BC × cotang. A) $= \frac{yy}{1-yy}$, and $1-yy=y$, or $yy+y=1$, whence $y = \frac{\sqrt{5}-1}{2}$.

Again, in the ſame triangle, 1 : coſ. AC : : coſ. BC : coſ. AB, whence coſ. AB = coſ. AC × coſ. BC; that is, $\sqrt{1-xx} = \sqrt{1-yy} \times \sqrt{1-yy} = 1-yy$, therefore $\sqrt{1-xx} = y$. And ſince AC =BC, therefore $\angle B = \angle A$. Hence all theſe five are equal; 1. Plain's declination. 2. Diſtance of the ſubſtile from the meridian. 3. Stile's height. 4. Latitude of the place. 5. Comp. of the plane's diff. longitude; and each of them = twice the ſine of 18° = .618034; whence the latitude and declination = 38° : $10'\frac{1}{3}$.

PROB. CXLVIII.

Given the ſun's meridian altitude, and alſo his altitude at two; to find the latitude.

Let Z be the zenith, P the pole; B, O two 91. places of the ſun. Let a, b = ſine and coſine of BZ (PO — PZ); x, y = ſine and coſine of PO + PZ, c = coſ. P, d = coſ. ZO. Then (Trig. I. 6. ſchol.) $ay+bx$ =S.PO, $by-ax$ = coſ. PO, $ay-bx$ =S.PZ, $by+ax$ = coſ. PZ. And in the triangle OPZ (Trig. III. 38.) $\overline{ay+bx} \times \overline{ay-bx} \times c = \overline{by-ax} \times \overline{by+ax} = d$; that is, $caayy - cbbxx + bbyy - aaxx = d$, but $yy = 1 - xx$, therefore $caa - caaxx - cbbxx + bb - bbxx - aaxx = d$, but $aa+bb=1$, whence $caa - cxx + bb - xx = d$, and $xx - cxx = caa + bb - d$, and

Fig. 91. $x = \sqrt{\frac{caa + bb - d}{1 - c}}$. Whence PO (or PB), and PZ are had.

PROB. CXLIX.

92. *In the ſpherical triangle* VAC, *there is given the perpendicular* AB, *the angle* A, *and the baſe* VC; *to find the ſegments.*

Put $b =$ S.AB, $t =$ tang. VAC, $c =$ tang. VC, $x =$ tang. BC. In the triangle BAC, $b : 1 :: x : \frac{x}{b} =$ tang. BAC, and (Trig. I. 9.) $1 + cx : 1 :: c - x : \frac{c - x}{1 + cx} =$ tang. VB, and $1 + \frac{tx}{b} : 1 :: t - \frac{x}{b} : \frac{tb - x}{b + tx} =$ tang. VAB. And in the triangle VAB, $1 : b :: \frac{tb - x}{b + tx} : \frac{c - x}{1 + cx}$. Whence $\frac{c - x}{1 + cx} = \frac{tb - x}{b + tx} b$. And multiplying, $bc - bx + tcx - txx = tb^2 - bx + bbctx - cbxx$, and reduced, $\overline{t - bc}.xx + \overline{bbct - tc}.x = bc - tbb$.

PROB. CL.

Travelling in an unknown part of the world, I found by chance an old horizontal dial, whoſe hour lines were ſo decayed by length of time, that I could only diſcover thoſe of 4 *and* 5; *whoſe diſtance I found juſt* 21 *degrees; to find the latitude of the place.*

Let $b =$ tang. 60 the hour arch of 4.
$d =$ tang. 75 the hour arch of 5.
$t =$ tang. of their difference 21.
$x =$ S. latitude.

Then by the known proportion of dialling, rad. : S. lat. :: tang. hour arch : tang. hour angle, that is, Fig.

$1 : x :: b : bx =$ tang. hour $\angle$ of 4.
$1 : x :: d : dx =$ tang. hour $\angle$ of 5.

But (Trig. I. 8.) $1 - btx : 1 :: t + bx : dx$, whence $t + bx = dx - bdtxx$, and $bdtxx + \overline{b - d}.x + t = 0$.

In numbers $2.48133x^2 - 2x = -.383864$.
and $x = .49084 =$ S.29 24 the lat.
or $x = .31518 =$ S.18 22 the lat.

PROB. CLI.

There are given the latitudes of three places lying in the arch of a great circle; and the diff. longitude are equal, between the middle one and each of the extream places; to find their diſtances.

Let P be the pole, AE the equinoctial; G, V, M, the three places. Put $b =$ tang. MD, $c =$ tang. VB, $d =$ tang. CG, $x =$ S.AB, $y =$ S.CB or BD. Then (Trig. I. 5.) $x\sqrt{1-yy} + y\sqrt{1-xx} =$ S.AD; and (Trig. I. 6.) $x\sqrt{1-yy} - y\sqrt{1-xx} =$ S.AC. 93.

Then (Trig. III. 27. cor. 1.) $x : c :: x\sqrt{1-yy} + y\sqrt{1-xx} : b$; and $x : c :: x\sqrt{1-yy} - y\sqrt{1-xx} : d$. Whence $\frac{bx}{c} = x\sqrt{1-yy} + y\sqrt{1-xx}$, and $\frac{dx}{c} = x\sqrt{1-yy} - y\sqrt{1-xx}$. Then adding and ſubtracting theſe laſt equations; $\frac{bx + dx}{c} = 2x\sqrt{1-yy}$, and $\frac{bx-dx}{c} = 2y\sqrt{1-xx}$; by the former $\sqrt{1-yy} = \frac{b+d}{2c} =$ coſ. CB or BD. Hence y is known. Therefore in the triangles GPV, VPM, two

Fig. two ſides and the included angle are given, to find
93. GV, VM, the diſtances required.

Or thus,

(By Trig. III. 44. cor.) As rad : $\frac{1}{2}$ tang. PV : : tang. GC + tang. MD : coſ. GPV or MPV.

PROB. CLII.

94. *In the ſpherical triangle* ABC, *we have given the angles* ADC, CDB, BDA, *about the point* D; *and the triangle* ABC *itſelf*; *to find the diſtances* AO, BO, CO.

Let s, c be the ſine and coſine of A; p, q=ſine and coſ. B, a=S.CDB, f=S.CB, b=S.CDA, d=S.AC, b=coſ. AB, n=coſ. ADB, x, v=ſine and coſ. DAB; y, z = ſine and coſ. DBA.

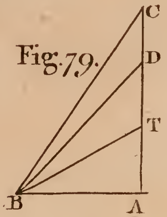
Then (Trig. I. 6.) $sv-cx$=S.CAD, and $pz-qy$=S.CBD, and in the triangles CAD, BAD, $b : d :: sv-cx : \frac{dsv-dcx}{b}$ = S.CD; and $a : f :: pz-qy : \frac{fpz-fqy}{a}$ = S.CD; therefore $adsv-adcx = bfpz-bfqy$.

In the triangle ADB (Caſe 10. Trig.) $bxy-vz=n$. But $v=\sqrt{1-xx}$, $z=\sqrt{1-yy}$, therefore $ads\sqrt{1-xx}-adcx=bfp\sqrt{1-yy}-bfqy$. and $bxy-\sqrt{1-xx}\times\sqrt{1-yy}=n$.

From theſe two equations, the roots may be eaſieſt found by problem xcv; otherwiſe it will aſcend to a high equation: or if you pleaſe you may proceed by rule 5, prob. xcii.

PROB.

Fig. 79.



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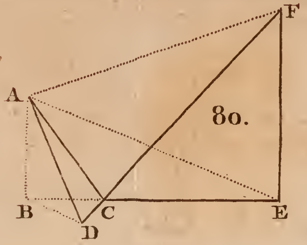

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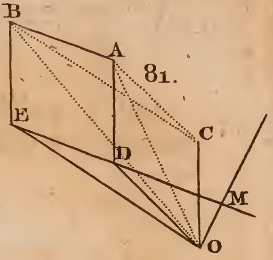

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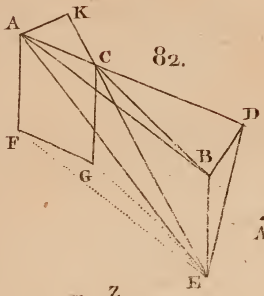

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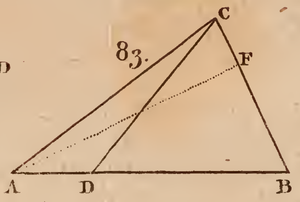

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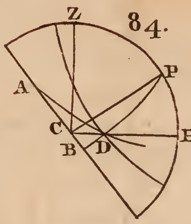

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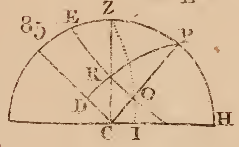

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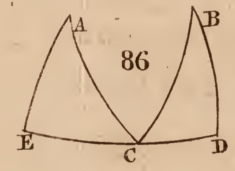

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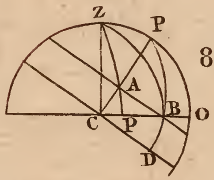

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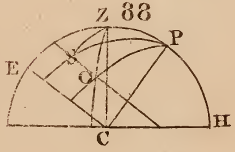

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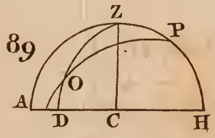

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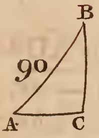

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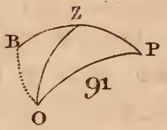

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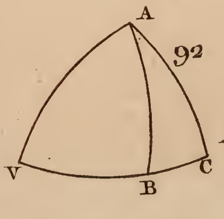

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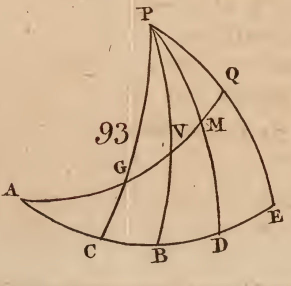

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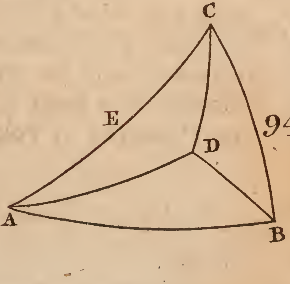

Pl. VIII. *pa.* 142

PROB. CLIII.

Fig.

Given the difference of the azimuths of three known ſtars; to find their altitudes.

Let Z be the zenith; A, B, C, the ſtars. Since 95. their places are given, the triangle ABC will be given. Put s, $c =$ ſine and coſ. $\frac{1}{2}$ABC, $a =$ S.AB, $b =$ S.BC, $n =$ coſ. AC, $q =$ coſ. AZC, $d =$ S.AZB, $e =$ S.CZB; $x, y =$ ſine and coſ. $\frac{\text{CBZ}-\text{ABZ}}{2}$. Then $sy + cx =$ S.CBZ $= v$, and $sy - cx =$ S.ABZ $= z$. And in the triangles ABZ, CBZ, $d : a :: z : \frac{az}{d} =$ S.AZ, and $b : e :: v : \frac{ve}{b} =$ S.CZ; and in the triangle AZC (Trig. III. 38.) $\frac{qazve}{db} + \sqrt{1-zz} \times \sqrt{1-vv} = n$; and tranſpoſing, $\sqrt{1-zz} \times \sqrt{1-vv} = n - \frac{qazve}{bd}$; and ſquaring, $1 - zz - vv + vvzz = nn - \frac{2nqae}{bd}vz + \frac{qqaaee}{bbdd}v^2z^2$; and tranſpoſing, $1 - nn = zz + vv - \frac{2nqae}{bd}vz + \frac{qqaaee-bbdd}{bbdd}v^2z^2$, but $z^2 + v^2 = 2ssy^2 + 2ccxx$, and $vz = ssyy - ccxx$. Therefore $1 - nn = 2ssyy + 2ccxx - \frac{2nqaess}{bd}yy + \frac{2nqaecc}{bd}xx + \frac{qqaaee - bbdd}{bbdd} \times \overline{s^4y^4 - 2s^2c^2x^2y^2 + c^4x^4}$. Let $1 - nn = mm$, $2 - \frac{2nqae}{bd} = f$, $2 + \frac{2nqae}{bd} = g$, $\frac{qqaaee}{bbdd} - 1 = p$: then $mm = fssyy + ccgxx + ps^4y^4 - 2pc^2s^2y^2x^2 + pc^4x^4$, but $yy = 1 - xx$; therefore expunging y, and reducing, $px^4 + \overline{gcc - fss - 2pss} \times xx = mm - fss - ps^4$.

PROB.

PROB. CLIV.

Given the ſun's declination, the difference of two altitudes, the difference of azimuths, and the difference of times; to find the altitudes, and the latitude.

96, Let P be the pole, Z the zenith; A, B, the places of the ſun. Put $d =$ S. ſun's declination, $s =$S.$\frac{1}{2}$APB, $c =$coſ. $\frac{1}{2}$AZB; p, $q =$ſine and coſine $\frac{\text{BZ}-\text{AZ}}{2}$; x, $y =$ ſine and coſ. $\frac{\text{BZ}+\text{AZ}}{2}$; then (Trig. I. 6. ſchol. 2.) $qx + py =$ S.ZB, $qx - py =$ S ZA, $qy - px =$coſ. ZB, $qy + px =$coſ. ZA. Then in the triangle API, $1 : d :: s : ds =$ S. half AD, and (Trig. I. 2. ſchol.) $1 - 2ddss =$ coſ. AB; and in the triangle AZB (Trig. III. 38.) $\overline{qqxx - ppyy} \times c + qq\ y - ppxx = 1 - 2ddss$; but $pp + qq = 1$, and $xx + yy = 1$; therefore $cqqxx - cpp + cppxx + qq - qqxx - ppxx = 1 - 2ddss$, or $cqqxx + cppxx - qqxx - ppxx = 1 - 2ddss + cpp - qq$; that is, $cxx - xx = 1 - 2ddss + cpp - 1 + pp = cpp + pp - 2ddss$, whence $x = \frac{\overline{c+1}.pp - 2ddss}{c-1}$. Then the ſides ZA, ZB are known. In the triangle PAI, find the <A; and in the triangle ZAB, find the angle A, and their difference ZAP; then in the triangle ZAP, there's given two ſides and the included angle A; to find ZP, the co-latitude.

PROB. CLV.

Given two altitudes of the ſun or a ſtar, and the times of obſervation; to find the declination, and latitude.

Let Z be the zenith, P the pole; B, A, the
97. places of the ſun or ſtar. Let $c =$ coſ. BZ, $f =$

f=coſ. AZ, b=coſ. ZPB, d=coſ. ZPA, x=coſ. BP − ZP, y = coſ. BP + ZP. Fig. 97.

Then (Trig. III. 42. cor. 1.) $x-y:2::x-c:1-b$, and $x-y:2::x-f:1-d$; therefore $x-c:1-b::x-f:1-d$; therefore $x-dx-c+cd=x-f-bx+bf$; and $bx-dx=c-f+bf-cd$, whence $x=\frac{bf-cd+c-f}{b-d}$. Alſo $\overline{1-b}\times\overline{x-y}=2x-2c$, or $by-y=x+bx-2c$, and $y=\frac{x+bx-2c}{b-1}=\frac{2c}{1-b}-\frac{1+b}{1-b}x=\frac{bf-cd+f-c}{b-d}$.

PROB. CLVI.

Given two altitudes of the ſun, the difference of times, and difference of azimuths; to find the latitude and declination.

Suppoſe P the pole, Z the zenith; B, A, the places of the ſun. Put c=coſ. BZA, s=coſ. BPA, y=S.BP or PA; b, d=ſine and coſ. ZB; f, p = ſine and coſ. ZA. 98.

Then (Trig. III. 38.) in the triangle BZA, $bcf+dp$ = coſ. BA; and in the triangle BPA, $syy+\sqrt{1-yy}\times\sqrt{1-yy}$ = coſ. BA; therefore $syy+1-yy=bcf+dp$, and $\overline{s-1}.yy=bcf+dp-1$, and $y=\sqrt{\frac{1-bcf-dp}{1-s}}$ = S.BP the declination.

Then in the triangle BPA, find the angle B, and in the triangle ZBA find the angle B, and then the difference ZBP. Then in the triangle ZBP there are given two ſides ZB, BP, and the included angle B; to find ZP, the co-latitude.

Fig.

PROB. CLVII.

Given the ſun's declination, two altitudes, and the time between them; to find the latitude.

98. Let a, b=ſine and coſ. PA; d, f=ſine and coſ. PB; n=coſ. ZB, m=coſ. ZA; s, c=ſine and coſine $\frac{1}{2}$BPA; y, z = ſine and coſ. $\frac{ZPA + ZPB}{2}$; v, x = ſine and coſ. ZP.

Then (Trig. I. 6. ſchol.) $cz - sy$ = coſ. ZPA and $cz + sy$ = coſ. ZPB, and (Trig. III. 38.) $aczv - asyv + bx = m$, and $dczv + dsyv + fx = n$; and $cz - sy = \frac{m-bx}{av}$, and $cz + sy = \frac{n-fx}{dv}$, and by adding and ſubtracting, $2cz = \frac{n-fx}{dv} + \frac{m-bx}{av} = \frac{an-afx+dm-dbx}{adv}$, alſo $2sy = \frac{n-fx}{dv} - \frac{m-bx}{av} = \frac{an-afx-dm+dbx}{adv}$. Whence $z = \frac{an+dm-afx-dbx}{2cadv}$, and $y = \frac{an-dm+dbx-afx}{2sadv}$. Put $\frac{an+dm}{2cad} = p$, $\frac{an-dm}{2sad} = q$, $\frac{db+af}{2cad} = r$, $\frac{db-af}{2sad} = t$. Then $z = \frac{p-rx}{v}$, and $y = \frac{q+tx}{v}$. But $z = \sqrt{1-yy}$, and $v = \sqrt{1-xx}$; then $\sqrt{1-yy} = \frac{p-rx}{z}$, and $1 - yy = \frac{\overline{p-rx}|^{2}}{vv}$, and tranſpoſing $yy = 1 - \frac{\overline{p-rx}|^{2}}{vv} = 1 - xx - \frac{\overline{p-rx}|^{2}}{vv}$. And $yy =$

$q + tx$

$\frac{\overline{q+tz}^2}{vv}$. Whence $1-xx-pp+2prx-rrxx=qq+2qtx+ttxx$; and reduced Fig. 98.

$$\begin{array}{llll} ttxx & +2tqx & +qq & =0. \\ +rr & -2pr & +pp & \\ +1 & & -1 & \end{array}$$

Or thus,

In the triangle BPA, 2 ſides and the included angle are given, to find $<$B, and BA. In the triangle BZA, all the ſides are given, to find the angle B, and from thence ZBP. Then in the triangle ZPB, two ſides and the included $<$B, are given; to find ZP the comp. of the latitude.

PROB. CLVIII.

Given three altitudes of the ſun in one day, and the times between them; to find the latitude, &c.

Let s, c = ſine and coſ. APB; t, b = ſine and coſ. ABC, d = coſ. AZ, f = coſ. BZ, g = coſ. CZ, x = S.ZPA, y = S.AP, BP or CP; z = S.ZP. 99.

Then (Trig. I. 5. cor. 1.) $c\sqrt{1-xx}-sx$ = coſ. ZPB, and $b\sqrt{1-xx}-tx$ = coſ. ZPC. And (Trig. III. 38.) $zy\sqrt{1-xx}+\sqrt{1-zz}\times\sqrt{1-yy}=d$, $czy\sqrt{1-xx}-szyx+\sqrt{1-zz}\times\sqrt{1-yy}=f$, and $byz\sqrt{1-xx}+tyzx+\sqrt{1-zz}\sqrt{1-yy}=g$. By tranſpoſition, $\sqrt{1-zz}\times\sqrt{1-yy}=d-zy\sqrt{1-xx}$. Then $czy\sqrt{1-xx}-szyx+\overline{d-zy\sqrt{1-xx}}=f$, and $bzy\sqrt{1-xx}-tzyx+\overline{d-zy\sqrt{1-xx}}=g$. From the former of theſe two laſt equations we get $zy=\frac{d-f}{1-c\sqrt{1-xx}+sx}$; and from the latter $zy=\frac{d-g}{1-b\sqrt{1-xx}+tx}$; and making theſe laſt equations

Fig. 99. tions equal, and reducing, we have $\frac{x}{\sqrt{1-xx}} = \frac{\overline{d-f} \times \overline{1-b} - \overline{d-g} \times \overline{1-c}}{\overline{d-g} \times s - \overline{d-f} \times t}$ = tang. ZPA. Then x will be known, and alſo zy.

But $\sqrt{1-zz} \times \sqrt{1-yy} = d - zy\sqrt{1-xx}$. Put $zy = r$, then $1 - yy - zz + zzyy = dd - 2dzy\sqrt{1-xx} + zzyy - zzyyxx$, and tranſpoſing, $yy + zz = 1 - dd + 2dr\sqrt{1-xx} + rrxx = p$ by ſubſtitution; then $yy + 2yz + zz = p + 2r$, and $yy - 2yz + zz = p - 2r$, and $y + z = \sqrt{p - 2r}$, and $y - z = \sqrt{p - 2r}$, whence $y = \frac{\sqrt{p + 2r} + \sqrt{p - 2r}}{2}$, and $z = \frac{\sqrt{p + 2r} - \sqrt{p - 2r}}{2}$.

PROB. CLIX.

Having at one inſtant the altitudes of two known ſtars; to find the latitude.

100. Let Z be the zenith, P the pole; F, A, the ſtars. In the triangle APF, there is given the ſides AP, FP the co-declinations, and angle P, the diff. right-aſcenſions, to find AF, and ∠F. Then in the triangle ZFA, all the ſides are given, to find the angle F; then ∠ZFP will be known. Let c = coſ. ZFP; a, b = ſine and coſ. ZF; d, f = ſine and coſ. FP; x = coſ. ZP. Then (Trig. III. 38.) $adc + bf = x$ = S. latitude.

PROB. CLX.

If there be two known ſtars in one azimuth, and having the altitude of either given; to find the latitude of the place.

101. Let Z be the zenith, P the pole; F, A the two ſtars in the azimuth circle ZFA, and ZF is

given. In the triangle APF, two ſides and the included angle P, are given; to find the angle F. Put c=coſ. <ZFP; a, b=ſine and coſine of ZF; d, f=ſine and coſine of FP; x = coſ. ZP. Then in the triangle ZFP (Trig. III. 38.) $adc + bf = x$ the ſine of the latitude. Fig. 101.

If ZA is given, you muſt find the angle FAP, and put a, b = S. and coſ. ZA; d, f = S. and coſ. AP, &c.

Otherwiſe thus,

Let the altitude of A be given; s, c = ſine and coſ. APF; d, f = ſine and coſ. AP; m, n = ſine and coſ. FP; a, b = ſine and coſ. ZA, x = coſ. ZP. Then (Trig. caſe 7. ſpherical triangles) co-tang. A $= \frac{dn - mfc}{ms}$, and tang. A $= \frac{ms}{dn - mfc} = t$, by ſubſtitution; and (Trig. I. 1. ſchol.) coſ. A $= \frac{1}{\sqrt{1+tt}}$. And (Trig. III. 38.) $\frac{ad}{\sqrt{1+tt}} + bf = x$ the S. latitude.

This is a uſeful problem.

* * *

SECT. IX.

Geometrical Loci, and Problems relating thereto.

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102. IF the right line AP be drawn from a given point A, and any number of right lines PM, PM, *&c.* be drawn thereon, parallel to one another, or making any given angle with AP. And if the relation of the indetermined quantities AP, PM be denoted in general by ſome equation; and if the lengths of PM be every where, ſuch as that equation gives; then the curve paſſing through all the points M, is called the *Locus* of the points M, or the locus of that equation. And that equation declares the nature of the curve MM.

The *degrees* of the *Loci* are denominated from the degrees of the equations, by which they are denoted. Thus a locus, of the firſt degree is that where the indetermined quantities riſe to one dimenſion; of the ſecond degree, when they ariſe to two dimenſions; of the third degree, when they riſe to three dimenſions, *&c.*

Right lines are ſaid to be given in poſition, when they make given angles with one another.

PROB. CLXI.

103. *If the pole* AC *or* BC *revolves about the center* C, *and the weight* D, *and ſtring* BD *hangs at the end of it; to find the nature of the curve* GD, *deſcribed by the weight* D.

Take AG, and CF, = BD. Then ſince CF, BD are equal and parallel, therefore (Geom. I. 5.

cor. 3.) CB, FD, are equal, or FD = CB = CA. Fig.
And ſince FC = GA, add CG, and then FG = CA; 103.
whence FD = FG. Therefore GD is a circle whoſe center is F, the ſame with AB, but in a lower poſition.

PROB. CLXII.

Suppoſe ACD, *acd*, &c. *to be right-angled triangles,* 104.
one of whoſe angles falls upon the fixed point A, *the other in the line* AE; *and if the ſegments* BD, *bd*, *be given*; *to find the nature of the curve paſſing through all the right-angles* C, *c*, &c.

Let AB = x, BC = y, BD = a; then in the right-angled triangle ACD; AB (x) : BC (y) : : BC (y) : BD (a); whence $ax = yy$, for the nature of the curve. Therefore the curve Cc is a parabola, whoſe latus rectum is BD or bd, and A the vertex.

PROB. CLXIII.

A *is a fixed point,* AB *a given line,* ABD *a given* 105.
angle; then ſuppoſe the curve AMB *to be generated after ſuch a manner that drawing any line* AC, *it may be, as* BC *to* BP : : *as* r : *to* s : *to find the nature of the curve, or the locus of all the points* M.

Draw MP, mp parallel to DB, and let AP = x, PM = y, AB = a. Then by ſimilar triangles AP (x) : PM (y) : : AB (a) : BC = $\frac{ay}{x}$. And by the problem, r : s : : BC $\left(\frac{ay}{x}\right)$: BP ($a - x$). Therefore $\frac{say}{x} = ra - rx$, and $y = \frac{r}{sa} \times \overline{ax-xx}$, for the nature of the curve; and it paſſes through A; becauſe when x is = 0, y is = 0.

Fig.

PROB. CLXIV.

106. *Given the triangle* ABC, *and drawing* PD *parallel to* BC; *ſuppoſe it be always* $PM^2 = PD^2 - BC^2$; *to find the nature of the curve* BM.

Put $AB=a$, $BC=b$, $AP=x$, $PM=y$, then by ſimilar triangles, $a : b :: x : \frac{bx}{a} = PD$, and by the queſtion, $\frac{bbxx}{aa} - bb = yy$, or $aayy = bbxx - bbaa$, and $y = \frac{b}{a}\sqrt{xx-aa}$. And the curve paſſes through B.

PROB. CLXV.

107. *The triangle* ABC *is given being right-angled at* B, *and drawing* FM *parallel to* CB, *it be every where* $PF^2 + PM^2 = BC^2$; *to find the nature of the curve* BM.

Let $AB = a$, $BC=b$, $AP=x$, $PM=y$. Then by ſimilar triangles, $a : b :: x : \frac{bx}{a} = PF$. And by the queſtion, $\frac{bbxx}{aa} + yy = bb$, whence $yy = bb - \frac{bbxx}{aa}$, and $y = \frac{b}{a}\sqrt{aa-xx}$, the equation for all the points M. And in A, where x is 0, $y = b$, or $AD = CB$.

PROB.

Fig.

PROB. CLXVI.

If AB, CF, AC *be right lines given in poſition*; *and* PD, *pd be always parallel to* AC; *and if* PM *be every where equal to* CD; *to find the locus of the point* M. 108.

Since AC, PD are parallel; AP will be to CD in a given ratio (Geom. II. 12. cor. 2.); put $AP=x$, $PM=y$, and let $a : b :: AP\ (x) : \frac{bx}{a} = CD$; therefore $y = \frac{bx}{a}$. Therefore AM*m* is a right line, paſſing through A.

PROB. CLXVII.

If the three lines CA, CB, AB *be given in poſition*; *and* PM *be always drawn parallel to* AB; *and it be every where* $AP \times PD = PM^2$; *to find the nature of the curve* M*m*. 109.

Let $CA=a$, $AB=b$, $AP=x$, $PM=y$; then by ſimilar triangles, $a : b :: a+x : \frac{a+x}{a}b = PD$, then by the queſtion, $\frac{a+x}{a}bx = yy$, and $y = \sqrt{\frac{b}{a} \times \overline{ax+xx}}$; and the curve paſſes through A, ſince both x and y are 0, at once.

But if CB be parallel to CA, then $PD = AB$, and $bx = yy$, or $y = \sqrt{bx}$. And if C lie on the other ſide of A, then $PD = \frac{a-x}{a}b$, and $y = \sqrt{\frac{b}{a} \times \overline{ax-xx}}$. In which caſe, when $x = a$, then $y = 0$, and the curve paſſes there through the axis C.

When

Fig. 109. When x is greater than a, y is the ſquare root of a negative quantity, which is impoſſible; and therefore the curve goes no further than A.

PROB. CLXVIII.

110. *There is given the right-angled triangle* ABD; *and drawing* PM, *pm always parallel to* BD; *and making* PM *every where equal to* BF; *to find the nature of the curve* DM*m*.

Put AB $=a$, BD $=b$, BP $=x$, PM $=y$. Then by ſimilar triangles, $a : b :: a+x : \frac{a+x}{a}b$ $=$ PF, and BF^2 $= xx = \frac{bb \times \overline{a+x}|^2}{aa} = \frac{aaxx + bbaa + 2bbax + bbxx}{aa}$; therefore $y = \sqrt{bb + \frac{2bbx}{a} + \frac{aa+bb}{aa}xx}$, for the nature of the curve.

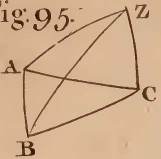
PROB. CLXIX.

111. BA *is a given line, draw* PM *perpendicular to* BAP; *and let* AM *be always a mean proportional between* AB *and* AP; *to find the nature of the curve* A*m*M.

Let AB $=a$, AP $=x$, PM $=y$, then AM $= \sqrt{xx+yy}$, and *per* queſt. $a : \sqrt{xx+yy} :: \sqrt{xx+yy} : x$, and $ax = xx + yy$, whence $y = \sqrt{ax - xx}$.

PROB.

Fig. 95.



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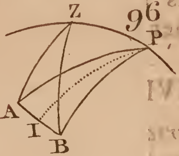

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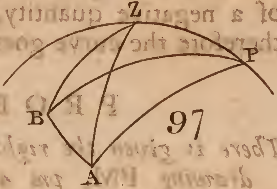

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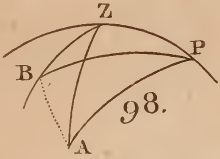

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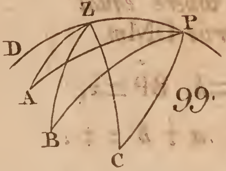

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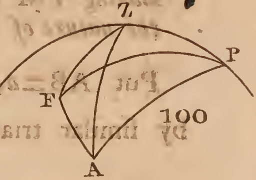

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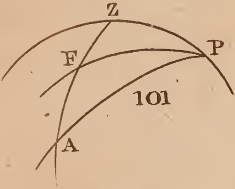

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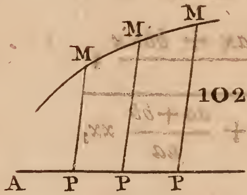

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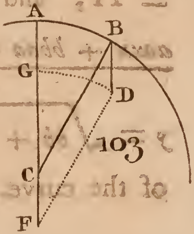

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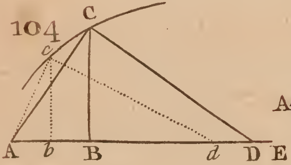

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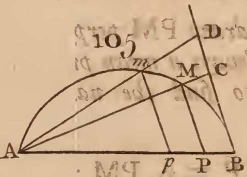

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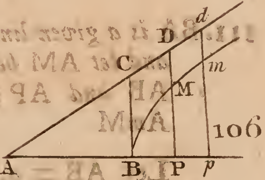

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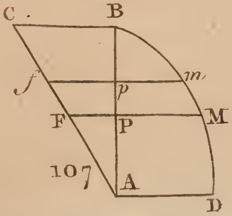

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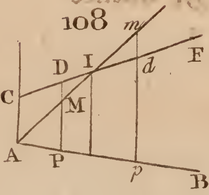

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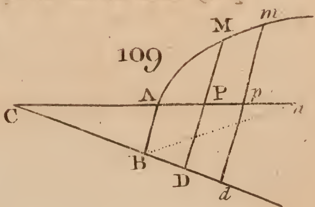


Pl. IX. pa. 454

PROB. CLXX.

The line CF *and point* A *being given; from any point* D *in that line, through* A, *draw* DAM, *and make* DA × AM *always equal to a given ſquare; to find the locus of* M. 112.

Draw BAP perpendicular to CF, and PM perpendicular to AP, and put $AB = a$, $AP = x$ $PM = y$, then $AM = \sqrt{xx+yy}$, $bb =$ the given ſquare. By ſimilar triangles $x : \sqrt{xx+yy} : : a : \frac{a\sqrt{xx+yy}}{x} = AD$. Then *per* queſt. $\frac{a}{x} \times \overline{xx+yy} = bb$, and $xx+yy = \frac{bbx}{a}$, whence $y = \sqrt{\frac{bbx}{a} - xx}$.

PROB. CLXXI.

AD *is a circle,* C *its center; draw* AB *perpendicular to* CAP. *Then draw any line* CB, *and* BM *parallel to* AP; *and make always* BM = DB; *to find the nature of the curve* mM. 113.

Draw MP perpendicular to CP, and let $CA = r$, $AP = x$, $PM = y$, then $CB = \sqrt{rr+yy}$, and $BD = \sqrt{rr+yy} - r$. But AP or BM = BD, that is, $x = \sqrt{rr+yy} - r$, and $\sqrt{rr+yy} = r + x$, and ſquaring, $rr + yy = rr + 2rx + xx$, whence $y = \sqrt{2rx + xx}$.

Fig.

PROB. CLXXII.

114. *CAD is a given angle,* CD *a given line,* M *a given point in it. Let this line ſo move in the angle* CAD, *that the ends* D, C *may always touch the ſides* AD, AC; *to find the curve deſcribed by the point* M.

Draw MP, MB parallel to AC, AD, and put DM $= a$, CM $= b$, coſ. ∢ A $= c$, AP $= x$, PM $= y$. By ſimilar triangles, CM (b) : BM (x) : : DM (a) : DP $= \frac{ax}{b}$. But in the triangle MPD. (Trig. caſe 5.) $yy + \frac{aaxx}{bb} - \frac{2acxy}{b} = aa$ or $yy = aa + \frac{2acxy}{b} - \frac{aaxx}{bb}$, for the equation of the curve.

PROB. CLXXIII.

115. *The line* CA *is perpendicular to* AP, ABM *is a ſquare whoſe ſides* CB, BM *are given; to find the curve deſcribed by the point* M, *whilſt the* ∢ B, *and end* C, *move along* PA, AC.

Draw MP perpendicular to AP, CB$=a$, BM$=b$, AP $= x$, PM $= y$; the triangles CAB, BMP are ſimilar, and $b : y :: a : \frac{ay}{b} =$ AB. And BP $= \sqrt{bb - yy}$, therefore $\frac{ay}{b} + \sqrt{bb-yy} = x$, and $\sqrt{bb-yy} = x - \frac{ay}{b}$, and ſquaring $bb - yy = xx - \frac{2axy}{b} + \frac{aavy}{bb}$. And reduced $\frac{aa+bb}{bb}yy = bb + \frac{2ayx}{b} - xx$.

PROB.

Fig.

PROB. CLXXIV.

The lines AV, AB, AE *are given in poſition, the point* V *is given; if the line* VE *be always drawn through* V, *and the part intercepted* CE *be divided in a given ratio at* M; *to find the locus of the points* M, *m*. 116.

Draw MP parallel to AB, and DM, BE parallel to AP, and put $AP = x$, $PM = y$, $AV = a$. r to s as CM to ME, S. $<BAE = p$, S. $<EAP$ or $BEA = q$.

The triangles VAC, VPM, DCM, BCE are ſimilar, and $a + x : y :: a : \frac{ay}{a+x} = AC$, and $a + x : y ::$ (DM) $x : \frac{xy}{a+x} = DC$. Alſo $r : s ::$ CM : ME :: CD $\left(\frac{xy}{a+x}\right)$: $DB = \frac{sxy}{ra + rx}$. And CM : CE $:: r : r + s :: x : \frac{r+s}{r} x = BE$, and in the triangle BAE, $p : q ::$ BE $\left(\frac{r+s}{r} x\right)$: $BA = \frac{r+s}{pr} qx = y + \frac{sxy}{r \times \overline{a+x}}$, and $\overline{r+s} \times qx \times \overline{a+x} = pry \times \overline{a+x} + psxy$, which reduced is (putting $\overline{r+s} = d$) $dqxx + dqax = pray + dpxy$, and $y = \frac{dqxx + dqax}{pra + dpx} = \frac{dq}{p} \times \frac{ax + xx}{ar + dx}$.

PROB. CLXXV.

BCD *is a given angle,* D *a fixed point,* BM *parallel to* CP; *and* BM *to* MD *are always in a given ratio; to find the locus of* M. 117.

Draw MP parallel to BC; and put $CA = a$, $AD = b$, coſ. $<P = c$, $AP = x$, $PM = y$; then PD =

Fig. $PD = b - x$, and $BM = a + x$; then (Trig. caſe 5.)
117. $MD = \sqrt{yy + \overline{b-x}\vert^{2} - 2cy \times \overline{b-x}}$. Whence it is, $a : b :: a+x : \sqrt{yy + \overline{b-x}\vert^{2} - 2cy \times \overline{b-x}}$; which ſquared and multiplied, $bbaa + 2bbax + bbxx = aayy + aabb - 2aabx + aaxx - 2caaby + 2caayx$, and reduced $aayy + 2caayx - 2caaby = bbxx + aaxx + 2bbax + 2aabx$.

PROB. CLXXVI.

118. *C, D are two fixed points in the line CD; and CM ſquare is every where to MD ſquare, in the given ratio of r to s. To find the locus of M.*

Draw MP perpendicular to CD, and let $CA = a$, $AD = d$, $AP = x$, $PM = y$, then $CP = a + x$, $PD = b - x$. Therefore it is $r : s :: aa : bb ::$ $(CM^{2})\ yy + \overline{a+x}\vert^{2} : (MD^{2})\ yy + \overline{b-x}\vert^{2}$; therefore $bbyy + bbaa + 2bbax + bbxx = aayy + aabb - 2aabx + aaxx$. Whence $aayy - bbyy = bbxx - aaxx + 2bbax + 2aabx$; and $yy = \frac{a+b}{aa-bb} \times 2abx -$ $xx = \frac{2ab}{a-b} x - xx$.

PROB. CLXXVII.

119. *The lines AB, AD are given by poſition; the points P, B, and angles CPD, CBD are given. Now if the angles CPD, CBD move about the centers P, B, whilſt the interſection D (of the ſides PD, BD) runs along the line AD; to find the curve which the interſection C, of the other ſides, deſcribes.*

Draw CS, DF perpendicular to AB; and put $AP = a$, $Pb = b$, tang. $<PAD = t$, tang. $CPD = p$,

tang. CBD $= q$, PF $= v$, PS $= x$, SC $= y$, and Fig.
BF $= b - v$, BS $= b - x$. 119.

Then by Trigonometry, $1 : a+v :: t : ta+tv$ $=$ DF.

And $v : 1 :: ta+tv : \frac{ta+tv}{v} =$ tang. DPF.

Alſo $x : 1 :: y : \frac{y}{x} =$ tang. CPS.

And $b-v : 1 :: ta+tv : \frac{ta+tv}{b-v} =$ tang. DBF.

And $b-x : 1 :: y : \frac{y}{b-x} =$ tang. CBS.

But (Trig. I. 8.) $1 - \frac{ta+tv}{vx} y : 1 :: \frac{ta+tv}{v} + \frac{y}{x} : p.$

And $1 - \frac{ta+tv}{b-v} \times \frac{y}{b-x} : 1 :: \frac{ta+tv}{b-v} + \frac{y}{b-x} : q.$

and multiplying the extreams and means,

$$pvx - ptay - ptvy = tax + tvx + vy$$

And $q \times \overline{b-v} \times \overline{b-x} - taqy - tqvy = \overline{ta+tv} \times \overline{b-x} + by - vy$; that is, $qb \times \overline{b-x} - qv \times \overline{b-x} - taqy - tqvy = tab - tax + tv \times \overline{b-x} + by - vy$; and tranſpoſing, $\overline{tv+qv} \times \overline{b-x} + tqvy - vy = qb \times \overline{b-x} - taqy + tax - tab - by$. By this and the former equation, $v = \frac{qbb - qbx + taqy - tab + tax - by}{qb - qx + tqy + tb - tx - y} = \frac{tax + ptay}{px - tx - pty - y}$. And ſubſtituting for the known compound quantities,

$$\frac{cx+dy+f}{-gx+hy+l} = \frac{tax+tapy}{nx-sy}\text{; reduced}$$

$$\begin{array}{llllll} taphyy & - tagxx & - tapgxy & + talx & + taply & = 0. \\ + sd & - cn & + tab & - fn & + sf & \\ & & + sc & & & \\ & & - dn & & & \end{array}$$

PROB.

Fig.

PROB. CLXXVIII.

120. *To find the figure for the ſection of a groin, the baſes of the two ſolids being* AFL *a ſemicircle, and* ABC, *a right-angled iſoceles triangle.*

Groining in joinery is fitting two priſmatic ſolids to join at right angles, ſo that the ſurfaces of both may coincide, no part of one being higher than the other, and the ends of both of them muſt be cut away to a certain figure, or elſe they can never join truly.

Let the perpendicular ſections AFL, ABC of the ſolids be perpendicular to the plane LACD, on which the figure is to be drawn. And ſuppoſe AMD to be the figure; draw MI, MP parallel to AC, AL; at I, P, draw the ordinates IF, PO, perpendicular to AL, AC. Now the nature of the groin requires that the lines FI, and PO, which are to coincide, muſt be equal. Therefore compute FI, OP in both figures, and put them equal to one another.

Let AL or $AC=a$, $AP=x$, $PM=y$. Then $IF=\sqrt{AI\times IL}=\sqrt{\overline{a-y}\times y}$; and ſince ABC is a right angle and $AB=BC$, OP will $=AP$; therefore $OP=x$, whence $x=\sqrt{ay-yy}$. Whence AM is an arch of a circle equal to AF. And for the ſame reaſon, the part at D of AMD is a like arch, and the whole curve AMD conſiſts of two quadrants of the circle AFL, meeting in the middle, and turning contrary ways. Therefore if the ends of the two ſolids, be cut into the figures ELDMAF, and BCDMAB; they will exactly fit one another.

PROB.

Fig.

PROB. CLXXIX.

To find the figure of a groin, when the baſes or ends of the bodies are AFL *a ſemicircle, and* ABC *the ſegment of a circle.* 121.

Let AMD be the curve; draw MP, MI parallel to AL, AC; and IF, PO perpendicular to AL, AC.

Put AL$=a$, AC$=b$, AP$=x$, PM$=y$. Then becauſe the figures APL and ABC muſt always be of equal height, therefore $\frac{1}{2}a =$ the height of ABC. Then to find the diameter of ABC, we ſhall have $\frac{1}{4}bb+\frac{1}{4}aa$ divided by $\frac{1}{2}a$, for the diameter; put D = diameter, then D $-\,a =$ the diſtance of the chord AC from the center, put D $-\,a = c$, and PO or IF $= v$. Then by the nature of the circle (in the figure ABC), $2cv + vv = bx - xx$; and $ay - yy = vv$, in the figure AFL. Therefore $2cv+ay-yy=bx-xx$, and $2cv=yy+bx-xx-ay$, alſo $v = \sqrt{ay-yy}$, and $2cv=2c\sqrt{ay-yy}$ $yy -$ $ay + bx - xx$; which ſquared and reduced gives an equation of the fourth power for the *locus* of M.

PROB. CLXXX.

To find a general equation to the ellipſis, referred to any line as an axis; which ellipſis will therefore be the locus of that equation.

Let BFDG be the ellipſis, C the center: Let 122.
the point A be given, and any line AL, given in poſition, for the axis. Take the angle KAL at pleaſure, and through C, draw the diameter BD, parallel to AK, and FCG the conjugate to it, and AN, PM, LK parallel to FG. Put BC or CD $= t$,

Fig. 122. CD $=t$, $p=$ parameter belonging to BD, AL $=a$, LK $=b$, AK $=c$, CN $=f$, AN $=n$, AP $=x$, PM $=y$.

By the ſimilar triangles ALK, API; $a:b::x:\frac{bx}{a}=$ PI; and $a:c::x:\frac{cx}{a}=$ AI. Then PR $=n+\frac{bx}{a}$, RM $=y-n-\frac{bx}{a}$, CR $=\frac{cx}{a}-f$, BR $=t-f+\frac{cx}{a}$, RD $=t+f-\frac{cx}{a}$. And by the property of the ellipſis $2t:p::$ BRD : RM$^2::tt-ff+\frac{2cfx}{a}-\frac{ccxx}{aa}:yy-2ny+nn+\frac{bbxx}{aa}-\frac{2bxy}{a}+\frac{2bnx}{a}$; and multiplying extreams and means, and reducing,

$$\begin{array}{llllll} 2aatyy-4abtxy & +2tbbxx & -4aatny & +4abtnx & +2aatnn & =0 \\ & +pcc & & -2apcf & -paatt & \\ & & & & +paaff & \end{array}$$

An equation to the ellipſis FD referred to the axis AL. Where note, yy and xx have the ſame ſign. And if xy is in the equation, the ſquare of half its coefficient is leſs than the coefficient of xx multiplied by the coefficient of yy. And if xy be wanting, xx and yy have the ſame ſign.

PROB. CLXXXI.

123. *To find a general equation to the hyperbola, referred to any line as an axis; and which hyperbola will conſequently be the locus of that equation.*

Let DM be a hyperbola, C the center, AL any line drawn from the given point A. Make LAK any given angle; and through C draw the diameter BD, parallel to AK, and FCG its conjugate,

jugate, and draw AN, PM, LK parallel to FG. Fig.
Put BC or CD $=t$, $p=$parameter of BD, AL $=a$, 123.
LK $=b$, AK $=c$, CN $=f$, AN $=n$, AP $=x$,
PM $=y$.

From the ſimilar triangles ALK, API, we ſhall get (as in the laſt problem,) PI $=\frac{bx}{a}$, AI $=\frac{cx}{a}$, whence PR $=n+\frac{bx}{a}$, RM $=y-n-\frac{bx}{a}$, and CR $=\frac{cx}{a}-f$, and DR $=\frac{cx}{a}-f-t$, and BR $=\frac{cx}{a}-f+t$. And by the nature of the hyperbola, $2t : p :: (\mathrm{BR}\times\mathrm{DR})\ \frac{ccxx}{aa}-\frac{2cfx}{a}+ff-tt : (\mathrm{MR}^2)\ yy-2ny+nn-\frac{2bxy}{a}+\frac{2bnx}{a}+\frac{bbxx}{aa}$. And the means and extreams multiplied, and then reduced,

$$\begin{array}{llllll} 2taayy & -4btaxy & +2tbbxx & -4tnaay & +4tnbax & +2taann = 0. \\ & & -pcc & & +2acpf & -paaff \\ & & & & & +paatt \end{array}$$

Note, when xy is not in the equation, yy and xx have different ſigns. And if xy be there, the ſquare of half its coefficient is greater than the coefficient of xx multiplied by the coefficient of yy.

SECT.

Fig.

SECT. X.

Mechanical Problems.

PROB. CLXXXII.

124. *If the weight* P *break the beam* DE, *when ſupported looſe at* A, B; *to find what weight will break it, when the ends* D, E, *are fixed, that they cannot riſe.*

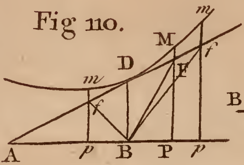
SUPPOSE DA = AC, and BE = BC. Suppoſe the beam cut through at C, and let $\frac{1}{2}$P be laid upon D, whilſt $\frac{1}{2}$P remains at C; then the preſſure at A will be = P, therefore the beam will alſo break at A, having the ſame ſtreſs there as it had at C. For the ſame reaſon, if $\frac{1}{2}$P be applied to E, CE will break at B. Conſequently, if 2P be applied to C, the beam being whole; and the ends D, E fixed; the beam will break at A, C, and B; and therefore it bears twice the weight of 2P, at C, before it breaks.

PROB. CLXXXIII.

125. *The ſtrength of a beam* AB, *being given; to find its ſtrength when a hole* (*ac*) *is cut out of the middle, and alſo an equal one* (*rn*) *in the ſide.*

By the principles of mechanics, (Mechan. 4to. Prop. LXX. and Cor.) the ſtrength of the beams whoſe thickneſſes are db, da, dc, will be as db^2, da^2, and dc^2. Now as the ſtrength of all the particles between b and d, is denoted by db^2, and the ſtrength of all the particles between a and

Fig 110.



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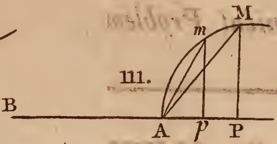

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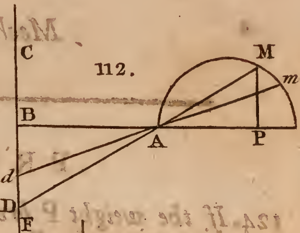

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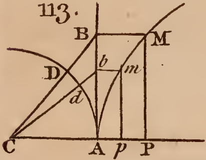

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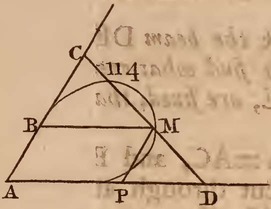

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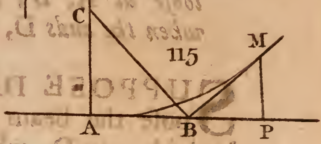

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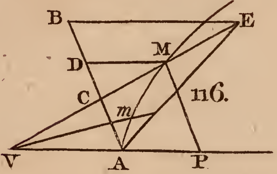

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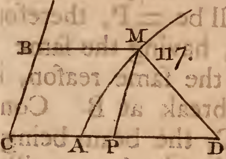

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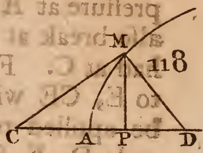

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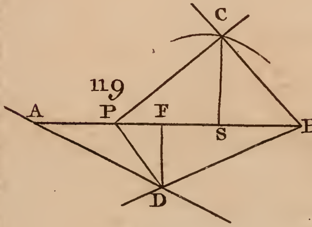
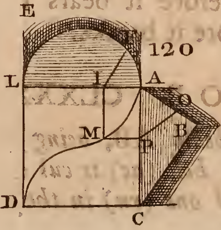


120



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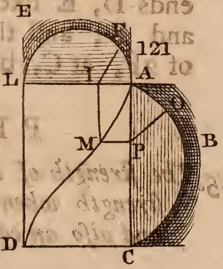

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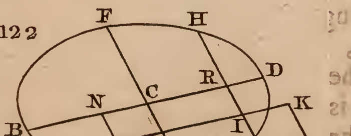
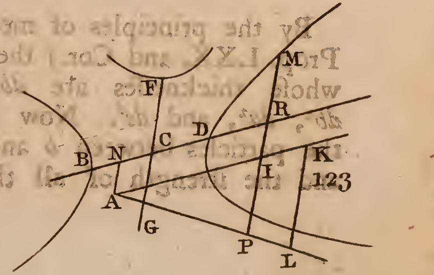




123

Pl.X.*pa.*464

and d, by ad^2; therefore the ſtrength of all the Fig.
particles between b and a, (the point D being 125.
fixed) will be $db^2 - da^2$, add the ſtrength between c and d, which is cd^2; and the ſtrength of ba and cd, that is, the ſtrength of the hollow beam is $db^2 - da^2 + cd^2$. But at the ſection r the ſtrength is fn^2.

Whence if $nr = ac$, the ſtrength at b to the ſtrength at r is as $db^2 - da^2 + cd^2$ to $\overline{db - ca}|^2$; that is, as $db^2 - 2dc \times ca - ca^2$ to $db^2 - 2db \times ca + ca^2$. Therefore if db^2 be the ſtrength of the whole beam, $\overline{2dc + ca} \times ca$ will be the defect of ſtrength of the hollow beam, when it breaks at b; and $\overline{2db - ca} \times ca$, the defect of ſtrength when it breaks at n or f, which is greater than the former. And for the ſame reaſon the defect of ſtrength to break at d, will be $\overline{2ba + ac} \times ca$.

P R O B. CLXXXIV.

To ſupport a long priſmatic body horizontally by two 126.
props A, B; *that it ſhall as ſoon break in* A *or* B *as in* C.

Let $DA = AF = GB = BE = y$, $CF = CG = x$, $DC = CE = n$, then $n = 2y + x$.

The parts AF and BG lay no ſtreſs upon C, being balanced by the contrary weights DA, BE, equal to them. Therefore the ſtreſs at C, ariſes from the weight FG; and muſt be equal to the ſtreſs at A, ariſing from the weights AD, AF.

The ſtreſs at A by the weight DF is $\frac{1}{2}DF \times DF$ or $2yy$, (Mechan. 70. and cor.) and the ſtreſs (by FG ſuſpended) at C is $AB \times FG$, or $\overline{2y + 2x} \times 2x$. But (ib. cor. 5.) $2AC$ $(2y + 2x)$: $AF + AC$ $(2y + x)$ :: ſtreſs at C, by FG ſuſpended at C $(\overline{2y + 2x} \times 2x)$: to the ſtreſs at C, in the poſition $FG = \overline{2y + x} \times 2x$.

Fig. 126. Therefore $2yy = \overline{2y+x}\times 2x$. Or $yy = 2xy + xx = nx = n\times\overline{n-2y}$, and $yy + 2ny = nn$. Whence $y = n\times\sqrt{2}-1$. And $x = n\times\overline{3-2\sqrt{2}}$.

PROB. CLXXXV.

127. *If two weights* P, T *keep one another in equilibrio, on the two wheels whoſe radii are* AB, CB; *the ſtrait tooth* AB *of the one acting on the crooked tooth* BD *of the other; to find the proportion of the weights* P, T.

Draw EBF perpendicular to OD, EH perpendicular to AB, and FG perpendicular to BC. The point B of the end AB, is acted upon by three forces: 1. in direction AB; 2. in direction BE; 3. in direction EH by the weight P; and theſe forces are as BH, BE, EH.

Again, the point B of the tooth BD is acted on by theſe three forces: 1. in direction BC; 2. in direction FB; 3. in direction FG by the weight T, and theſe forces are as BG, BF and FG. But the action and reaction at the point B, being equal; we have $BE = BF$, and in the right-angled triangle BHE, rad. (1) : EB : : S.ABE : $HE = EB \times$ S.ABE. And in the triangle BGF, rad. (1) : BF or EB : : S.FBG : $GF = EB \times$ S.FBG. Whence P : T : : HE : GF : : EB × S.ABE : EB × S.FBG; that is, P : T : : coſ. ABD : coſ. CBD, when the weights are in equilibrio.

Whence if ABC is a right line, $P = F$; and if $\angle CBD = 0$, then P : T : : coſ. ABO : radius.

PROB. CLXXXVI.

To find proper numbers for the wheels and pinions of a clock, to go eight days; and to ſhew hours by the great wheel, minutes by the ſecond wheel, ſeconds by the balance wheel, and to beat ſeconds.

For the moving part.

Suppoſe four wheels in the moving part A, B, C, D, and let the numbers for the wheels and pinions be denoted as in the figure, and let $f = 12$, $h =$ height the weight deſcends, $t =$ time of going down in hours. 128.

It is plain $\frac{A}{p} =$ number of revolutions B has for one of A, and $\frac{B}{q} =$ number of revolutions C has for one of B; whence $\frac{AB}{pq} =$ number of revolutions C has for one of A. And likewiſe $\frac{ABC}{pqr} =$ number of revolutions D has for one of A.

Since the arbor of D carries an index to ſhew ſeconds, therefore $D = 30$, becauſe for every tooth there are two beats, and $2D = 60$.

Becauſe the arbor of B carries an index to ſhew minutes, and of A to ſhew hours; conſequently A goes about in 12 hours, and B in 1, whence $\frac{A}{p} = 12$. And becauſe D goes 60 times round for B's once, therefore $\frac{BC}{qr} = 60$.

Therefore the two equations $\frac{A}{p} = 12$, and $\frac{BC}{qr} = 60$, will reſolve the queſtion; which being

Fig. unlimited, many of them may be taken at pleaſure,
128. provided they be all whole numbers.

Suppoſe $r = 6$, $p = 8$, $q = 8$; then $A = 96$, and $BC = 6 \times 8 \times 60$, and if $B = 60$, then $C = 48$; or $B = 72$, and $C = 40$. It will be better if B and q, C and r be prime to one another.

To find the diameter of the wheel for the rope, it will be $t : h :: f : \frac{fh}{t}$ = circumference, and $\frac{fh}{3.1416t}$ = diameter.

For the ſtriking part.

Let L be the fly, K the warning-wheel, I the detent wheel, H the pin-wheel, G the great wheel, F the count-wheel, their teeth and pinions as in the figure; n = number of ſtriking pins, and there are 78 ſtrokes in 12 hours: F goes round in 12 hours, I goes round for every ſtroke of the clock.

Now $\frac{78}{n}$ = number of revolutions of H in 12 hours, and $\frac{FG}{ab}$ = number of revolutions of H in one of F, that is, in 12 hours; therefore $\frac{FG}{ab} = \frac{78}{n}$.

Again, I goes round n times for H's once, and therefore $\frac{H}{c} = n$. Therefore from theſe two equations $\frac{FG}{ab} = \frac{78}{n}$, and $H = cn$, all the requiſites may be found; but being unlimited moſt of the numbers may be taken at pleaſure, ſo as they be all convenient whole numbers.

Becauſe the pin in the warning-wheel muſt always come to the ſame place when the clock has ſtruck out, therefore $\frac{1}{d} =$ a whole number. L Fig. 128. and e may be any numbers, becauſe there is no phenomenon to be ſhewn by them.

The train or beats in an hour is $= \frac{ABC}{pqr} \times \frac{60}{12} = \frac{5ABC}{pqr}$. Suppoſe $n = 12$, $a = 6$, $b = 8$, $c = 6$; then $H = 72$, and $FG = \frac{78 \times 48}{12} = 4 \times 78$, therefore F may be $= 13$ and $G = 24$. But note $\frac{FG}{ab}$ may be put into one wheel or more as one pleaſes.

If the ſtring go about the axis of F, its diameter is found as in the other. But if it go round the axis of G, it muſt be made leſs in proportion as a to F. If one weight carry both parts, their diameters muſt be but half the former quantities.

PROB. CLXXXVII.

Suppoſing with Borelli (part. I. prop. 22. de motu animalium), *that a ſtrong man can but bear* 26 lb. *at arm's end, and that the weight of his whole arm is equivalent to* 4 lb. *at arm's end; from the length of his arm given; to find the dimenſions of that man's arm, that can bear no more than its own weight.*

Suppoſe 4 *lb.* at arm's end equivalent to 8, the weight of the arm. And ſuppoſe the two arms, ſimilar ſolids, and the arm = half the length of the body. Put $a =$ length of a common man's arm, $b = 4$ *lb.* $w = 26$*lb*, $x =$ length of the great man's arm.

The weight of like bodies are as the cubes of the ſides, $a^3 : x^3 :: 2b : \frac{2bx^3}{a^3} =$ weight of the

Fig. great man's arm, and $\frac{bx^3}{a^3}$ = the weight at arm's end, producing the ſame ſtreſs.

And the ſtreſs being as the length and weight, we have $\overline{w+b} \times a$ = ſtreſs of the common man's arm; and $\frac{bx^3}{a^3} x$ = ſtreſs of the great man's arm. But (by mechanics) the ſtreſs in this caſe, is as the ſtrength, that is, as the cubes of the like ſides, Therefore $a^3 : x^3 :: \overline{w+b} \times a : \frac{bx^4}{a^3}$, whence $bx^4 = \overline{w+b} \times ax^3$, or $x = \frac{w+b}{b} a = \frac{30a}{4} = 7\frac{1}{2}a$. Now if $a = 1$ yard; then if there be a man whoſe height is above 15 yards; he will not be able to ſtretch out his arm.

PROB. CLXXXVIII.

129. *Given the length and poſition of the beam* AD, *leaning againſt the wall* DE; *to find the poſition of the plane* BE, *on which it may ſtand without moving.*

Let G be the center of gravity of the beam together with any weight it carries. Through G, draw the horizontal line BH. And ſuppoſe DA put into the poſition *da*, infinitely near the former. Now ſince the beam is to have no inclination of moving from the poſition DA, or *da*; the center of gravity G, *g* muſt be in the horizontal line BH, by the principles of mechanics. Draw G*n*, *dm*, A*r* perpendicular to *ad* or AD. And let DG $= b$, AG $= c$, h = S.DHG, p, q = ſine and coſ. ADH; s, f = ſine and coſine DGH, x = tang. DAE, v = DF.

Since DG = *dg* = *mn*, and AG = *ag* = *rn*, therefore D*m* = *ng* = *ar*. In the triangle D*dm*, S*md*D

(*q*)

(q) : S.mDd (p) : : mD : md $= \frac{p}{q} \times mD$ or $\frac{p}{q}$ Fig. 129. $\times gn$. And in the triangle Ggn, S.gGn (f) : S.Ggn (s) : : gn : Gn $= \frac{s}{f} \times gn$. By the similar triangles Fdm, FGn, Fd (v) : FG $(b - v)$: : ma $\left(\frac{p}{q} \times gn\right)$: nG $\left(\frac{s}{f} \times gn\right)$, whence $\frac{sv}{f} = \frac{pb - pv}{q}$, and $qsv = pbf - pfv$, and $v = \frac{pbf}{pf + qs}$. But (Trigon. I. 5.) $pf + qs = b$. Therefore $v = \frac{pbf}{b}$. In the triangle Aar, 1 : ar or ng : : x : rA $= x \times ng$. And in the similar triangles FDm, FAr, Fd (v) : md $\left(\frac{p}{q} \times gn\right)$: : FA $(b + c - v)$: rA $(x \times ng)$; therefore $vx = \frac{p}{q} \times \overline{b + c - v}$, and $vqx = pb + pc - pv$, and $vqx + pv = pb + pc$, and substituting the value of v, $\overline{qx + p} \times \frac{pbf}{b} = pb + pc$, and $bfqx + bfp = bb + cb$, whence $x = \frac{bb + cb - bfp}{bfq} = \frac{b + c}{bfq} b - \frac{p}{q}$. Whence

1. If DH be perpendicular to the horizon, $b = 1$, $s = q$, $f = p$, and $x = \frac{b + c}{bpq} - \frac{p}{q}$.
2. If DH nearly coincides with DA, $b = s$, $p = 0$, $q = 1$, then $x = \frac{b + c}{b} \times \frac{s}{f}$, or $\frac{b + c}{b} \times$ tang. DGH.

PROB. CLXXXIX.

Having given, the ſpecific gravity of two things, and likewiſe the ſpecific gravity of a mixture of them; to find the proportion of the things mixed.

Let A, B be the two things, and M the mixture, a, b, m, the ſpecific gravity of A, B, M; A, B, M their magnitudes. Then ſince the abſolute weight is as the magnitude and ſpecific gravity; therefore aA, bB, mM will be the weight of A, B, M. And $aA+bB=mM=m\times\overline{A+B}$, and tranſpoſing $aA-mA=mB-bB$. Whence $\overline{m-b}:\overline{a-m}::A:B$.

PROB. CXC.

Having given the weights and velocities of two ſpherical bodies perfectly elaſtic, meeting one another in a right line; to determine their velocities, after reflexion.

Let A, B, be the weights of the bodies, a, b, their velocities towards different parts, x and y their velocities the contrary way, after reflexion. Then Aa, Bb are the quantities of motion in their reſpective directions, before reflexion; and Ax, By after. As the bodies are elaſtic, they will recede from one another, with the ſame relative velocity they met, whence $a+b=y+x$. And (by mechanics) the difference of the motions, moving the ſame way, will remain the ſame after as before the ſtroke, therefore $Aa-Bb=By-Ax$, but $y=a+b-x$, therefore $Aa-Bb=Ba+Bb-Bx-Ax$; and tranſpoſing, $Ax+Bx=aB+bB-Aa+bB$, and $x=\dfrac{\overline{B-A}\times a+2Bb}{A+B}$, and $y=\dfrac{\overline{A-B}\times b+2Aa}{A+B}$.

PROB. CXCI.

ACDB is a thread fixed at A *and* B, *at the points* C, D *of this thread are fixed the two threads* CE, DF, *with the weights* E, F; *having given the weight* F, *and the poſition of the points* C, D, *to find the weight* E.

Let the weight $F = w$, weight $E = x$, $S.\angle CDB = s$, $S.FDB = t$, $S.DCA = p$, $S.ECA = q$. 130.

The point D is kept in equilibrio by 3 forces in directions DB, DC, DF, which are to one another, as the ſines of the angles they paſs through (Mechan. 8. cor. 2.): therefore S.CDB (s) : force at F (w) :: S.FDB (t) : force in DC $= \frac{tw}{s} =$ force in CD, becauſe action and reaction are equal and contrary.

Again, the point C is drawn by three forces, in directions CD, CA, CE; therefore, S.ECA (q) : force CD $\left(\frac{tw}{s}\right)$ :: S.ACD (p) : force at E (x), therefore $qx = \frac{ptw}{s}$, and $x = \frac{pt}{qs}w$.

PROB. CXCII.

Three points of the cieling, A, B, C *are given, to which are fixed the threads* AF, BF, CF *whoſe lengths are given; to theſe is fixed the thread* FD, *with the given weight* D; *to find the tenſion of all the threads.* 131.

Becauſe the triangle ABC is given, and the lengths of the threads; the point O will be given, where DF produced cuts the cieling. Produce AO to E, and draw EF, which will be $= \sqrt{FO^2 + OE^2}$.

All

Fig. 131. All the ſides of the triangles CFE, EFB, are given, and conſequently the angles. Now inſtead of the threads FC, FB, ſuppoſe the thread FE to ſuſtain the weight. And then the whole is ſuſtained by the two threads AF, FE acting in the perpendicular plane AOEF. Draw OL parallel to AF, in the plain AEF, and LG parallel to CF in the plane CFB.

Put $AE = a$, $AF = b$, $EO = c$, $AO = d$, $EF = f$, $OF = h$, $S.\angle CFB = p$, $S.CFE = q$, $S.EFB = s$.

Then (Mechan. 8.) the tenſion of the threads DF, AF, EF, will be denoted by OF, OL, LF; and taking away the thread FE, the tenſion of the threads CF, BF, will be LG, GF. Then to find each. By ſimilar triangles, EA (a) : AF (b) :: EO (c) : $OL = \frac{bc}{a}$. And EA (a) : AO (d) :: EF (f) : $LF = \frac{fd}{a}$. And in the triangle FLG, S.LGF (p) : LF $\left(\frac{fd}{a}\right)$ :: S.LFG (s) : $LG = \frac{sfd}{ap}$, and :: S.FLG (q) : $FG = \frac{qdf}{pa}$.

Therefore the tenſions of DF, AF, CF, BF, are reſpectively as h, $\frac{bc}{a}$, $\frac{sfd}{ap}$, $\frac{qfd}{ap}$.

SECT.

SECT. XI.

Philoſophical or Phyſical Problems.

PROB. CXCIII.

Required the height of the tower, from the top of which a ſtone falling to the bottom, the ſound will reach the ear at the top, in the time of the fall.

PUT $b=16\frac{1}{12}$ feet, the height a body falls in a ſecond.

$c=1142$ feet, the ſpace ſound moves through in a ſecond.

$a=$ time of the body's falling.

Then $1 : c :: a : ca=$ ſpace ſound moves in the time a.
And $1 : aa :: b : baa=$ height the ball deſcends.

Therefore *per* qu. $baa=ca$, and $a=\frac{c}{b}=71$ ſeconds.

And $baa=ca=\frac{cc}{b}=81088$, the height.

PROB. CXCIV.

There is a round tower, whoſe circumference is 100 *yards, a ſpiral tube runs about, from bottom to top, at an elevation of* 61° : 5′. *A ball put in at the top of this tube will run down to the bottom in* 8 *ſeconds; to find the height.*

Let $\angle$ABD be 61° : 5′, AC perpendicular to 132.
BD, and BC perpendicular to AB. Then whilſt a body falls through AC, another would deſcend through

Fig. through AB in the ſame time (Mechan. Prop. 34.
132. cor. 1. 4to.) Put $b = 16.1$ feet, $d = 8''$, $s = \text{S.ABD}$. Then by the laws of falling bodies, $1 : b :: dd : bdd$ = height fallen in $8'' = \text{AC}$. And rad. $(1) : \text{AC}\ (bdd) :: \text{S.C}\ (s) : bdds = \text{AB}$. And rad. $(1) : \text{AB}\ (bdds) :: \text{S.ABD}\ (s) : bddss = \text{AD}$, the height required $= 789$.

PROB. CXCV.

Given the diſtance of the earth and the moon, and their quantities of matter; to find the place where a body will be attracted to neither of them.

Let d = diſtance of their centers, l = matter in the moon, t = matter in the earth, x = diſtance from the earth where the body is, then $d - x$ = its diſtance from the moon.

Then ſince the force of attraction is as the matter directly, and the ſquare of the diſtance inverſely; therefore we have $\frac{t}{xx}$ = earth's attraction, and $\frac{l}{\overline{d-x}|^{2}}$ = moon's attraction; but *per* queſt. theſe are equal, therefore $\frac{t}{xx} = \frac{l}{dd - 2dx + xx}$; which reduced is $\overline{t - l}.xx - 2dtx + ddt = 0$.

PROB. CXCVI.

A clock that keeps true time on the ſurface of the earth; being carried to the top of a certain mountain, loſt 2 minutes in a day. What was the mountain's height?

Let r = earth's radius = 6982000 yards, $b = 1440$ minutes, $c = 2$ minutes, a = height of the mountain.

But (Mechan. prop. 40. cor. 6. 4to.) the length of a pendulum is as the force of gravity, and the ſquare of the time of vibration; and the length being given, the force of gravity is reciprocally as the ſquare of the time of vibration.

But the force of gravity is alſo as the ſquare of the diſtance from the earth's center; therefore the time of vibration of the ſame pendulum, is as the diſtance from the earth's center: and the number of vibrations in a given time, reciprocally as that diſtance. Therefore $r : \frac{1}{b} :: r+a : \frac{1}{b-c}$, and $\frac{r+a}{b} = \frac{r}{b-c}$, and $r+a = \frac{br}{b-c}$. Whence $a = \frac{br}{b-c} - r = \frac{cr}{b-c} = 9697$.

PROB. CXCVII.

A ball projected from the top of a tower, at an elevation of 31 deg. above the horizon, did in $9\frac{1}{6}$ *seconds fall 2000 feet from its baſe; to find the height.* 133.

Let x=VB the tower's height, BA$=d$ the diſtance, b = tang. DVC = 31°, $t = 9\frac{1}{6}$ the time, $f = 16.1$; then

In the time t, the ball without gravity would arrive at D, and in the ſame time it would deſcend through DA. Whence $1 : f :: tt : ttf$ = DA by the laws of falling bodies.

And in the triangle DVC, $1 : b :: d : db$=DC, and DC+CA=DA, or $db + x = ttf$, and $x = ttf - db = 149$.

Fig.

PROB. CXCVIII.

If a ball be dropped from the top of a tower a mile high, on the ſide facing the eaſt, in latitude 51½; where will it fall?

134. Let the body fall at D, whilſt the tower by the rotation of the earth is carried to IC. Now by the laws of centripetal force, the area AIE, which the body, moving in the circle AIF deſcribes; is equal to the area AGDE, which the body moving in the curve AGD deſcribes in the ſame time, that is, in the time of falling through AB. Hence the area AGI=area EGD; and AGDF=EIF. But by reaſon of the ſmall diſtance BD, the curve AGD (which otherwiſe would be an ellipſis) is nearly a parabola; and the area of AFD$=\frac{1}{3}$AF×AB$=\frac{1}{2}$FI×AE, the area of the ſector EIF. Firſt, let A be a place in the equinoctial.

Put BE $= r =$ 21000000 feet, AB$=m=$5280; $f=16.1$, $t=$24 hours=86400″, $c=3.1416$, $a=$ DC, $p=$ coſ. 51½. Then by the laws of falling bodies, $\sqrt{f} : 1 :: \sqrt{m} : \sqrt{\frac{m}{f}}$ = time of falling through AB. And $t : 2rc :: \sqrt{\frac{m}{f}}$: BC $= \frac{2rc}{t}\sqrt{\frac{m}{f}} = d$, and BD $= a + d$; alſo by ſimilar ſectors, $r : r+m :: a+d : \frac{m+r}{r} \times \overline{a+d} =$ AF. And $r : a :: r+m : \frac{r+m}{r}a =$ FI. Therefore $\frac{r+m}{3r} \times \overline{a+d} \times m = \frac{r+m}{2r}a \times \overline{r+m}$; therefore $\overline{a+d} \times \frac{m}{3} = \frac{r+m}{2}a$, and $2am + 2dm = 3ra + 3ma$, and reduced $a = \frac{2dm}{3r+m} = 4.64$; and $pa =$ 2.88 for the lat. 51½.

Fig. 135.

PROB. CXCIX.

There are two iſlands A, C; *at* C *is a caſtle. A ſhip from* A *to* C *keeps pace with the waves of the ſea,* 100 *in number, from* A *to* B. *At* B *ſhe fires a gun, which ecchoes back from the caſtle to* B, *in* 3 *ſeconds; and the time of ſailing from* B *to* C *was* 3 *minutes; to find the diſtance* AC.

Let $b=100$, $c=3''$, $d=3'$, $f=39.2$ in. the length of a ſecond pendulum, $a=1142$ feet, the velocity of ſound in a ſecond, AC $=y$, and $x=$ breadth of a wave.

Then by the motion of pendulums, $\sqrt{f}:1::\sqrt{x}:\sqrt{\frac{x}{f}}=$ time of vibration of the pendulum x.

And $\sqrt{\frac{x}{f}}:x::1'':\sqrt{fx}=$ a ſpace. But (by the principles of philoſophy) while the pendulum x vibrates once, the ſhip or a wave runs through the breadth x; or in 1 ſecond runs through the ſpace $\sqrt{fx}$.

And $1'':\sqrt{fx}::d'':d\sqrt{fx}=$ CB.

Alſo by the motion of ſound $1'':a::\frac{1}{2}c:\frac{ca}{2}=$ CB, for the eccho returns with the ſame velocity the ſound went. Therefore $d\sqrt{fx}=\frac{ca}{2}$, and $ddfx=\frac{ccaa}{4}$, and $x=\frac{ccaa}{4ddf}$; therefore AB $=bx=\frac{bccaa}{4ddf}$, and $y=\frac{ca}{2}+\frac{bccaa}{4ddf}$.

PROB.

Fig. 136.

PROB. CC.

Suppoſing a planet and its ſatellite to move in circular orbits, it is required to find, whether the path of a ſatellite is concave or convex to the ſun, when it is in a line between the ſun and its primary.

At the time of the conjunction, if the planet and ſatellite, both deſcribe very ſmall arches in the ſame time, whoſe verſed ſines are equal, the ſatellite will then move in a right line. Let ABC be the orbit of the planet, EF that of the ſatellite; whilſt the planet moves through AB, the orbit of the ſatellite EF is moved into the poſition *ef*, and the ſatellite has moved from *e* to o. Draw BD, *oh* perpendicular to AE, B*e*; and CG perpendicular to AG. Now put AD = *ne*, then ſince the center (of the orbit EF,) A is advanced to B, nearer to the line CG, by the diſtance AD; and the point o is receded from the ſame line CG, by the diſtance *en* equal to AD; it is plain, E and *o* are equidiſtant from GC, and E*o* is a right line, or the ſatellite E, o, at that time moves in a right line.

Let $a = eo$, $b = AB$, r, s = the radii of *eo*, AB. Hence AD $= \frac{bb}{2s}$, and $en = \frac{aa}{2r}$, and $\frac{bb}{2s} = \frac{aa}{2r}$. Put p = periodic time of the ſatellite, q = that of the primary; $c = 3.1416 \times 2$. Then $cr : p :: a : \frac{pa}{cr}$ = time of deſcribing a; and $\frac{bq}{cs}$ = time of deſcribing b, then $\frac{pa}{r} = \frac{qb}{s}$, and $\frac{ppaa}{2rr} = \frac{qqbb}{2ss}$, divide this by the former equation, and $\frac{pp}{r} = \frac{qq}{s}$, or $pp = \frac{rqq}{s}$. Therefore as pp is greater, equal, or leſſer than $\frac{rqq}{s}$, then the ſatellite's orbit is concave, ſtreight, or convex towards the ſun, in its conjunction. Fig. 136.

PROB. CCI.

To find the diviſions of a monochord, to ſound all the half notes, according to equal intervals of ſound; and alſo to find the variations between theſe and the ſtrict harmonic diviſions.

It is well known an octave is divided into 6 whole tones, or 12 ſemitones Let BA be the monochord or vibrating ſtring, C the middle point; then BC will be an octave above BA. Let B*d*, B*e*, B*f*, B*g*, &c. be the ſeveral lengths of the ſtrings ſounding the half notes, gradually aſcending, above AB, by equal degrees of ſound. Then will A*d*, *de*, *ef*, &c. be all unequal in length; and whatever part B*d* is of BA, the ſame part will B*e* be of B*d*, and B*f* of B*e*, and B*g* of B*f*, &c. to make the ſeveral ſounds aſcend equally. Therefore BA, B*d*, B*e*, B*f*, &c. are a ſet of geometrical proportionals decreaſing, continued to 13 terms, the laſt of which is BC. Alſo A*d*, *de*, *ef*, &c. are a ſet of geometrical proportionals in the ſame ratio. 137.

Put $BA = 1$, $BC = \frac{1}{2}$, $Bd = x$. Then BA (1) : B*d* (x) : : B*d* (x) : $Be = xx$; likewiſe $Bf = x^3$, $Bg = x^4$, &c. and $BC = x^{12} = \frac{1}{2}$. And $x = \sqrt[12]{\frac{1}{2}} =$ $\cdot 9439$.

Or, put $X = \log : x$. Then $X = \frac{\log : \frac{1}{2}}{12} = -$ 1.9749142, conſequently 2X, 3X, 4X, &c. = logarithms of x^2, x^3, x^4, &c. Therefore $x =$

137. .9439, $x^2 = .8909$ for a mean tone, &c. and the reſt are as in the following table.

The harmonic diviſions of the monochord, to ſound the pure concords will be, as follows; the leſſer third $= \frac{5}{6}$, greater third $\frac{4}{5}$, fourth $\frac{3}{4}$, fifth $\frac{2}{3}$, leſſer ſixth $\frac{5}{8}$, greater ſixth $\frac{3}{5}$, eight $\frac{1}{2}$; which ſee in the following table, in decimals.

Names of the chords.	Pure concords.	Equal diviſions.	Errors.
whole ſtring	1.0000	1.0000	0.
b ſecond	.....	.9439	...
♯ ſecond	.....	.8909	...
leſſer third	.8333	.8409	*b* $\frac{1}{13}$
greater third	.8000	.7937	♯ $\frac{1}{15}$
fourth	.7500	.7492	♯ $\frac{1}{100}$
♯ fourth	.....	.7071	...
fifth	.6666	.6674	*b* $\frac{1}{100}$
leſſer ſixth	.6250	.6300	*b* $\frac{1}{15}$
greater ſixth	.6000	.5946	♯ $\frac{1}{13}$
b ſeventh	.....	.5612	...
♯ ſeventh	.....	.5297	...
Eight	.5000	.5000	0.

138. Then to find the errors or variation of the correſpondent cords. Let Bt = cord by column 2d, Br = cord by column 3d, rp = a whole tone, n = number of mean proportionals between Br and Bp, then $\frac{1}{n}$ will be the error, for it ſhews what part rt is of the whole note rp. Here then $\frac{Bt^n}{Br^{n-1}} = Bp$ $= Br \times .8909$. For .8909 being a whole note for

the ſtring 1, $Br \times .8909$ will be = note for the ſtring Br. Therefore $\frac{\overline{Bt}|^{n}}{Br} = .8909$. And $n \times \log : \frac{Bt}{Br} = \log : .8909 = -1.94983$; and $n = \frac{-1\ 94983}{\log:Bt-\log:Br} = \frac{.05017}{\log:Bt-\log:Br}$. As in a fifth, $\log:Bt-\log:Br = .000500$, and $n = \frac{.05017}{.000500} = 100$; whence the error $= \frac{1}{100}$.

Fig. 138.

But as this variation bears but a ſmall proportion to the length of the ſtring, there will be no need to make uſe of logarithms. For ſince $1-.8909=.1091$ is the length of a note when the ſtring is 1; therefore $.1091 \times Bt$ = a note for the ſtring Bt. Whence $\frac{rt}{rp}$ or $\frac{rt}{tp}$ = the error, or which is the ſame thing $\frac{Br-Bt}{.1091Bt}$ = the error. As in the fifth, $Br-Bt=.6674-.6666\frac{2}{3}=.0007$, and $.1091 \times Bt=.0727$, and $\frac{.0727}{.0007}=100$, nearly, or $\frac{7}{727} = \frac{1}{100}$ = the error.

Or ſhorter thus. Since $Br-Bt$ = twice the difference of two adjoining numbers in col. 3. or = difference of two numbers 2 degrees diſtant, taking one greater and the other leſs than the proper note; therefore $\frac{Br-Bt}{Br-Bp}$ = the error.

As in the fifth, $\frac{6674-6666}{7071-6300} = \frac{8}{771} = \frac{1}{97}$ the error. And in a greater third, $\frac{.8000-.7937}{.8409-.7492} = \frac{.0063}{.0917} = \frac{1}{14\frac{1}{2}}$ the error.

Fig. 138 The errors for each concord being thus computed, are ſet down in the fourth column, which ſhews the error of the third column, as it differs from the ſecond; thoſe below denoted by (♭), theſe above, by (♯).

In tuning a harpſichord, ſince the fifth muſt be 12 times repeated to make 7 octaves, therefore the variation, by tuning by true fifths, will be $\frac{12}{100}$ or about $\frac{1}{8}$ of a note, which is an error that a good ear can diſcover; and being too ſharp, the fifths therefore ought to be tuned as flat as the ear will bear.

Hence the equal diviſion of the notes in an octave is the beſt ſyſtem, for the greateſt error is in the leſſer third and greater ſixth, which only amounts to $\frac{1}{13}$ of a note.

PROB. CCII.

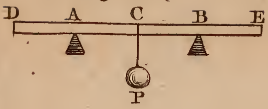
To find the number of beats made in any imperfect concord, in muſic.

I call that an *imperfect concord* that varies a little from the perfect one, which is made by a harmonial diviſion of the monochord. Thus when the lengths of the ſtrings are 4 and 5, you have the perfect cord (a greater third), but vary one length as 4, making it 3.99, and you will have an imperfect cord attended with beats.

A *beat* is a jarring ſound made by the irregular vibrations of two ſtrings, ſounding together, when the due period, or coincidence of their vibrations is interrupted. Its noiſe is ſuch as this *waw*, *aw*, *aw*, *aw*, or *yâ*, *yâ*, *yâ*, *yâ*, *yâ*, Our buſineſs is to find in how many vibrations this perturbation happens, or how many *yaws* in a ſecond of time.

Let

Fig. 124



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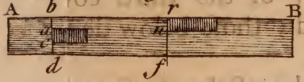
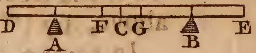
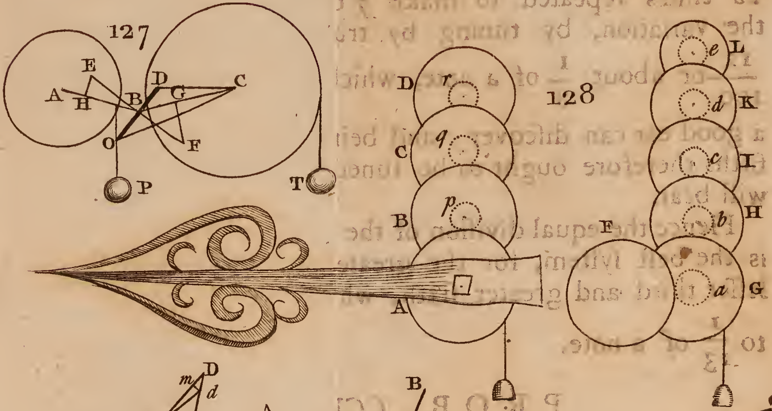


126



127

128



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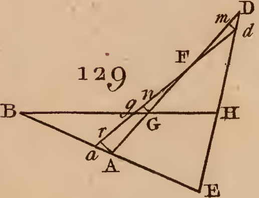

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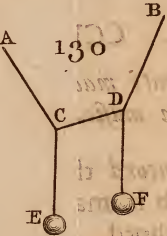
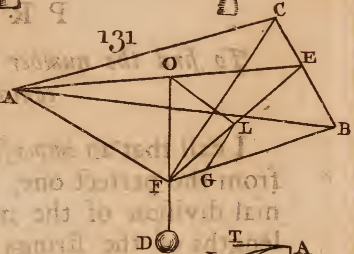


131



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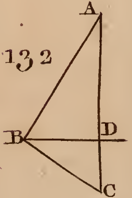

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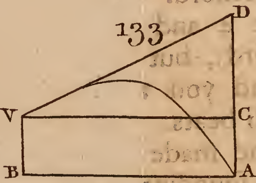

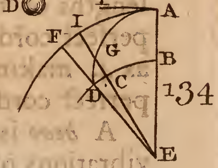

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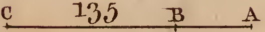

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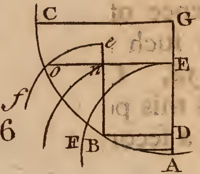

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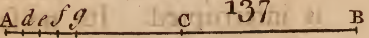

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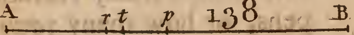


Pl. XI. pa: 484

Fig. 139. Let the line AZ repreſent one ſecond of time; and ſuppoſe it divided on the under ſide, into the number of vibrations of the lower note or baſe, at A, B, C, D, *&c.* and the upper ſide into the number of vibrations of the upper note, at *a*, *b*, *c*, *d*, &c. Now if any number of diviſions on the under ſide coincide with any number of diviſions on the upper, conſtantly and regularly, as at C and *d*, E and *g*, &c. Then the concord is pure, and there is no beat. But when the points *b*, *c*, *d*, &c. are any of them diſlocated, and gets to the other ſide of its correſponding one; then the ſucceſſion of the ſhort harmonic periods of coincidence is diſturbed; and this cauſes the noiſe called a beat, ſuch as happens at X and Y. For *c*, *f*, *i*, &c. are continually approaching to B, D, F, *&c*, till they fall in at X, Y, and change ſides; where B*c* or B*b*, is ſuppoſed the leaſt diſtance, in the firſt harmonical period AC, ſuppoſing *ad* was to coincide with AC. Therefore at the points X, Y, the ſucceſſion of the harmonical periods are confuſed, and that periodically,) which ſpoils the harmony.

Now to find the length of this period. Let AC be one harmonical period, that is, when *d* coincides with C, as in the pure concord. In this falſe cord we muſt find the time *d*C, which is gained or loſt in the time AC. And from thence compute in what time, B*c* (the neareſt diſtance), would be gained or loſt in; and that will be the time required.

Let n = number of parts AB, BC, *&c.* or its number of vibrations.
t = number of vibrations of the upper ſtring in the perfect cord.
c = length of its ſtring on the monochord.
r = number of parts *ab*, *bc*, *cd*, &c. or its number of vibrations.

Fig. 139. b = length of its ſtring on the monochord. $\frac{p}{q}$ = number expreſſing the concord, $= \frac{3}{4}$ for the fourth, or $= \frac{2}{3}$ for the fifth, &c.

Then $AB = \frac{1}{n}$, and $AC = \frac{p}{n}$; alſo $ab = \frac{1}{r}$, and $ad = \frac{q}{r}$. Then $AC - ad = \frac{p}{n} - \frac{q}{r} = dC$. Then if $dC \left(\frac{p}{n} - \frac{q}{r}\right)$ be loſt or gained in the time $AC \left(\frac{p}{n}\right)$: : AB will be loſt in the time $\frac{\frac{p}{n}}{\frac{p}{n} - \frac{q}{r}} \times AB = \frac{pr}{pr-qn} \times AB = \frac{pr}{pr-pt} \times AB = \frac{r}{r-t} \times AB$, and Bc will be loſt ſooner, in proportion of AB to Bc, that is, in the time $\frac{r}{r-t} \times Bc$, which is the time of the period. But by the laws of vibration, $r : t : : \frac{1}{b} : \frac{1}{c} : : c : b$, and $r-t : r : : c-b : c$; whence $\frac{r}{r-t} \times BC = \frac{c}{c-b} \times Bc$ = the periodic time of the beats. And if AZ be divided by the periodic time, you will have $\frac{c-b}{c} \times \frac{AZ}{Bc}$ = number of beats in a ſecond. But $Bc = \frac{AB}{q}$, $AZ = n \times AB$, therefore $\frac{AZ}{Bc} = \frac{nb \times AB}{AB} = nq$. Whence $\frac{c-b}{c} \times nq = \frac{c-b}{c} \times pt$ = number of beats in a ſecond.

Hence, from the length of the ſtring or diviſion Fig. 139. of the monochord, as given in the table of the laſt problem, and having alſo the number of vibrations; the beats will be found, as in this table. Where the ground or loweſt note is F the cliff-note of the baſe.

Cords.	Vibrations.	*b*	*c*	*p*. *q*.	Beats.
Eight	600	5000	5000	1 . 2	0
g. ſixth	500	5946	6000	3 . 5	13 #
l. ſixth	480	6300	6250	5 . 8	18 *b*
fifth	450	6674	6666	2 . 3	1 *b*
fourth	400	7492	7500	3 . 4	$1\frac{1}{3}$ #
g. third	375	7937	8000	4 . 5	11 #
l. third	360	8409	8333	5 . 6	15 *b*
Baſe F	300	10000	10000	1 . 1	0

This table ſhews the beats for all the concords, reckoning upwards from F; when the inſtrument is tuned according to an equal aſcent of notes; where the flats and ſharps (*b*, #) ſhew whether the upper note is lower or higher, than the true concord in the laſt column. In the octave above, the beats will be twice as many; and in the octave below, but half as many; being always proportional to the number of vibrations of the baſe note. The fifth is moſt ſerviceable in tuning, and the number of beats in one ſecond, for the fifth is $\frac{n}{300}$.

If it be ſuppoſed that the beat is not made at the points X, Y, but at ſome intermediate place, where they fall thicker and more confuſed; and that at the points X, Y, there is the leaſt imperfection. Yet the periodic time will ſtill be the ſame, whatever part of the cycle XY it falls in. When the cycle

Fig. 139. cycle XY is very ſhort, the ſingle beats are imperceptible, and we hear nothing but a diſagreeable noiſe. All the concords beat, but being exceeding quick, they are not perceived ſingly; and being regular throughout, they exhibit an agreeable harmony.

When the pitch of the two notes are not altered, the beats ſucceed one another in equal times, but altering either of them nearer to a perfect harmony, the beats ſucceed in longer times, and the nearer the longer, till at laſt they vaniſh, when the concord is perfect. All the beats are heard in organs; but only half of them are heard in ſtringed inſtruments.

* * *

SECT.

SECT. XII.

Problems relating to Series.

PROB. CCIII.

Given the diameter of a circle; to find the ſide of any regular poligon, inſcribed in it.

LET $d =$ diameter, $n =$ number of ſides, $x =$ ſide of the figure, EB. By Trigonometry, $\frac{3.14159d}{2n} =$ arch DE$=a$, by ſubſtitution. 140

And (Trig. I. 12.) half the ſide, or EA $= a - \frac{4aa}{6dd}A - \frac{4aa}{20dd}B - \frac{4aa}{42dd}C$ &c. And 2EA or EB or $x = 2a - \frac{4aa}{6dd}A - \frac{4aa}{20dd}B - \frac{4aa}{42dd}C - \frac{4aa}{72dd}D$ &c.

Or thus,

By a table of natural ſines, find the ſine of $\frac{180}{n} = s$, then $x = ds$.

PROB. CCIV.

Suppoſe $x^3 - cx^{\frac{1}{2}} + x^{\frac{2}{3}} + bx^{\frac{3}{4}} = dccx$; *to find* x.

Divide by the leaſt power of x, that is, by $x^{\frac{1}{2}}$, and $x^{\frac{5}{2}} - c + x^{\frac{1}{6}} + bx^{\frac{1}{4}} = dccx^{\frac{1}{2}}$. Take r the neareſt root, and put $r + e = x$.

Then

Fig. Then

$$\left.\begin{array}{rcl} x^{\frac{5}{2}} & = & r^{\frac{5}{2}} + \frac{5}{2}r^{\frac{3}{2}}e + \frac{15}{8}r^{\frac{1}{2}}ee \\ -c & = & -c \\ +x^{\frac{1}{6}} & = & r^{\frac{1}{6}} + \frac{1e}{6r^{\frac{5}{6}}} - \frac{5ee}{72r^{\frac{11}{6}}} \\ +bx^{\frac{1}{4}} & = & br^{\frac{1}{4}} + \frac{be}{4r^{\frac{3}{4}}} - \frac{3bee}{32r^{\frac{7}{4}}} \\ -dccx^{\frac{1}{2}} & = & -dccr^{\frac{1}{2}} - \frac{dcce}{2r^{\frac{1}{2}}} + \frac{dccee}{8r^{\frac{3}{2}}} \end{array}\right\} = 0.$$

That is, $p + qe + see = 0$, by ſubſtitution. Whence

$$e = \frac{-\frac{p}{s}}{\frac{q}{s} + e},$$ which may be repeated for more exactneſs.

Or thus,

Seek the leaſt common dividend of the denominators of the indices of x, and reduce the equation, which will become $x^{\frac{30}{12}} - c + x^{\frac{2}{12}} + bx^{\frac{3}{12}} - dccx^{\frac{6}{12}} = 0$. Put $x = e^{12}$; then the equation becomes $e^{30} - c + e^{2} + be^{3} - dcce^{6} = 0$, or $e^{30} - dcce^{6} + be^{3} + ee = c$, and the root extracted gives e, and conſequently x is had.

PROB. CCV.

141. *Given the ſides* AC, CB, *of the triangle* ACB; *and the ratio of* AB *to the arch* CE *is given*; *to find* AB.

Let AC $= r = 14$, CB $= s = 22$, AB $= x$, and AB : CE : : 10 : 4, whence arch CE $= \frac{4}{10}x$. Let $y =$ coſ. CAB.

Then (Trig. caſe 5.) $rr+xx-2rsy=ss$, whence $y=\frac{rr+xx-ss}{2rs}=$ coſ. A to the radius 1, and $ry=\frac{rr-ss+xx}{2s}=$ coſ. A to the radius r. But (Trig. I. 12. cor. 1.) coſ. $A=r-\frac{aa}{2r}+\frac{a^4}{24r^3}$ &c. $=r-\frac{\overline{4x}^2}{200r}+\frac{\overline{4x}^4}{24.10000r^3}-\frac{\overline{4x}^6}{720.1000000r^5}$ &c. $=\frac{rr-ss}{2s}+\frac{xx}{2s}$, and tranſpoſing, $\overline{\frac{1}{2s}+\frac{4^2}{200r}}\times x^2-\frac{4^4}{240000r^3}\times x^4+\frac{4^6}{720000000r^5}x^6$ &c. $=r-\frac{rr-ss}{2s}$, or $Ax^2+Bx^4+Cx^6$ &c. $=b$, by ſubſtitution, then (Prob. lxii. I.) $xx=\frac{b}{A}-\frac{B}{A^3}b^2+\frac{2BB-AC}{A^5}b^3$ &c. $=836.95$, and $x=28\,93$. Fig. 141.

PROB. CCVI.

Given the arch of the circle BHE, *and the ſine* BD *to find the radius* BC. 142.

Let BHE $=d=8$, BD $=s=3$. Take an angle p nearly equal to ACB, $a=$ ſine, $b=$ its coſine, rad. $=1.n=.0174533$, $c=3.1415926$, then $np=$ arch belonging to the angle p. Let $p+x=$ true angle ACB; then $np+nx$ or $np+z=$ correſpondent arch, (putting $z=nx$). And (Trig. I. 13.), the ſine of $np+z=a+bz-\frac{az^2}{2}-\frac{bz^3}{6}+\frac{az^4}{24}+$ &c. $=\frac{s}{d}c-\frac{s}{d}np-\frac{s}{d}z$ *per* queſt. that is, Fig. 142. $\overline{\frac{s}{d}+b}\,.\,z-\frac{a}{2}zz-\frac{b}{6}z^3+\frac{a}{24}z^4$ &c. $=\frac{s}{d}c-\frac{s}{d}np-a$, or $Az+Bzz+Cz^3+Dz^4$ &c. $=R$. Aſſume $p=55°$, then a and b will be known, and $R=-,0010292$, and (Prob. lxii. I.) z or $nx=\frac{R}{A}-\frac{B}{A^3}R^2+\frac{2BB-AC}{A^5}R^3$ &c. $=-.001084$, and $x=-0621$ degrees, which is $3'\,43''\frac{1}{2}$; therefore $p+x=54°\;56'\;16''\frac{1}{2}=<ACB$. Hence $BC=3.66513$.

PROB. CCVII.

143. *The ordinate* AC, *and curve* BC *are given; and the equation of the curve is* $\frac{y}{a}=$ *hyp. log:* $\frac{z+\sqrt{aa+zz}}{a}$ *and* $a+x=\sqrt{aa+zz}$; *where* $AC=y$, $BC=z$; *to find* AB.

Let $y=5=b$, $z=9=g$, $a=r+e$, taking r the neareſt value of a. Then hyp. log : $\frac{g+\sqrt{rr+2re+ee+gg}\text{ \&c.}}{r+e}=\frac{b}{r+e}=\frac{b}{r}-\frac{be}{rr}+\frac{bee}{r^3}$ &c. And (putting $ff=rr+gg$), by evolution, log: $\frac{g+f+\frac{re}{f}+\frac{gg}{f^3}ee}{r+e}=\frac{b}{r}-\frac{be}{rr}+\frac{bee}{r^3}$. and $\frac{f+g+\frac{re}{f}+\frac{gg}{f^3}ee}{r+e}=$ number of the hyp, log : $\frac{b}{r}-\frac{be}{rr}+\frac{be^2}{r^3}=n-\frac{nbe}{rr}+\frac{nbee}{r^3}+\frac{nbb}{2r^4}ee$ (putting $n=$ h. log : $\frac{b}{r}$) (Prob. lxxxv. I.)

And

Fig. 143.

And multiplying by $r+e$,

$$\begin{aligned} g+f+\frac{re}{f}+\frac{gg}{2f^3}ee = rn - \frac{nbe}{r} &+ \frac{nb^2ee}{2r^3} \\ + ne &+ \frac{nbee}{rr} \\ &- \frac{nbee}{rr} \end{aligned}$$

And reduced

$$\overline{\frac{r}{f}+\frac{nb}{r}-n} \times e + \overline{\frac{gg}{2f^3}-\frac{nbb}{2r^3}} \times ee = rn - f - g.$$

Aſſume $r = 3.7$; then $e = -.00112$, and $a = 3.69885$, and $x = \sqrt{aa+zz} - a = 6.0316$; ſubſtitute this value of a for r in the laſt equation, and the operation repeated, gives a ſtill more exact.

PROB. CCVIII.

Given the length of a pendulum, and the arch it deſcribes; to find the time loſt by deſcribing a greater arch.

Let r = length of the pendulum, c = cord of half the arch it deſcribes, C = any other cord, t = time of falling through $2r$. $P = \frac{3.1416}{2}t$. Then by mechanics it is found that the time of 1 vibration, is $= P \times : 1 + \frac{cc}{16rr} + \frac{9c^4}{4^5r^4}$ &c. for the cord c.

And $= P : \times 1 + \frac{CC}{16rr} + \frac{9C^4}{4^5r^4}$ &c. for the cord C.

and $P \times : \frac{CC-cc}{16rr} + \frac{9}{4^5r^4} \times \overline{C^4 - c^4}$ &c. = time loſt in one vibration for the cord c. But when

Fig. when c is 0, P is the time of one vibration, which does not ſenſibly differ from a vibration for the cord c. Therefore ſince 86400 = the number of ſeconds in 24 hours, therefore $\frac{86400}{P}$ = number of vibrations for the cord c, in 24 hours. Therefore $\frac{86400}{P} \times P \times \frac{CC-cc}{16rr}$ &c. or $86400 \times : \overline{\frac{CC-cc}{10rr} + \frac{9}{4^5r^4} \times \overline{C^4-c^4}}$ &c : = ſeconds loſt in 24 hours; that is, $\frac{5400}{rr} \times : \overline{\overline{CC-cc} + \frac{9}{64rr} \times \overline{C^4-c^4}}$ &c. = ſeconds loſt in 24 hours; and $\frac{5400}{rr} \times \overline{CC-cc}$, is nearly = the ſeconds loſt in 24 hours.

If r ſwings ſeconds, then $r = 39.2$, and the time loſt in 24 hours is nearly $= 3.52 \times \overline{CC-cc}$.

Cor. *If* c *be* = 0, *and* C = *cord of* 90°. *A pendulum vibrating in the double arch of* 90°, *will loſe* 4 *h.* 20 *min, in* 24 *hours time.*

And if $c = 0$, then to find the length of a pendulum vibrating in the arch of C in the ſame time. Let r = pendulum vibrating in the very ſmall arch, x = pendulum vibrating in the arch of C. Then the lengths being as the ſquares of the times of vibration, we ſhall have in the firſt caſe $\sqrt{r}$ for t, and in the ſecond $\sqrt{x}$ for t; whence in the firſt caſe $P = \frac{3.1416}{2}\sqrt{r}$, in the ſecond $P = \frac{3.1416}{2}\sqrt{x}$. And the times being equal we ſhall have $P \times 1$ or $\frac{3.1416}{2}\sqrt{r} = P \times$

1 +

$\overline{1+\frac{CC}{16rr}}$ or $\frac{3.1416}{2}\sqrt{x}\times\overline{1+\frac{CC}{16rr}}$, or $\sqrt{r}$ $=\sqrt{x}\times\overline{1+\frac{CC}{16xx}}$, and $r=x+\frac{CC}{8x}$, which reduced is $rx-xx=\frac{1}{8}CC$; whence x will be found. And on the contrary x being given, r will be found.

PROB. CCIX.

Given the latitude ſailed from, the departure, and difference of longitude; to find the difference of latitude.

Let $d=$ departure, $l=$ diff. longitude, $x=$ arch of latitude come to, $z=$ its mer. parts, $a=$ the given lat. $m=$ its mer. parts.

Then by *Mercator*'s Sailing; as diff. lat. $(a-x)$: mer. diff. latitude $(m-z)$ :: $d:l$, whence $al-lx=dm-dz$, and $dz-lx=dm-al$. But Dr. *Halley*'s Series for the meridional parts of x, is $x+\frac{1}{6}x^3+\frac{1}{24}x^5+\frac{61}{5040}x^7$ &c. Therefore $dx-lx+\frac{d}{6}x^3+\frac{d}{24}x^5+\frac{61d}{5040}x^7$ &c. $=dm-al$. And by reverſion of ſeries (Prob. lxii.) x will be found; then $3438x=$ latitude in minutes.

Or thus,

Seek another latitude, by the table of meridional parts, ſuch, that the proper difference of latitude divided by the mer. diff. latitude, will be equal to the quotient $\frac{d}{l}$, which is eaſily done by a few trials; and that is the other latitude.

PROB.

Fig.

PROB. CCX.

144. *The curve* BMD *is deſcribed with a pair of compaſſes upon the ſurface of a cylinder, which is afterwards ſtretched into a plane; to find the ordinate* PM.

Let $d =$ diameter of the cylinder, $a =$ AB the extent of the compaſſes, AP $= x$, PM $= y$, $v =$ cord, whoſe arch is y. Then (Geom. II. 21.) $aa - xx = vv$. But (Trig. I. 12. cor. 2.) $v = y - \frac{yy}{2.3dd} A - \frac{y^2}{4.5dd} B -$ &c. Whence $\sqrt{aa - xx} = y - \frac{yy}{2.3dd} A - \frac{yy}{4.5dd} B - \frac{yy}{6.7dd} C$ &c. and by reverſion of ſeries y is had.

If the arch was in a given ratio to the chord, the figure would be an ellipſis; but as this is not ſo, the curve will be a mechanical one.

* * *

SECT. XIII.

Problems concerning exponential quantities.

PROB. CCXI.

Some maids driving a flock of ſheep, were aſked, how many they had? To which they anſwered, that if the flock was equally divided among them, the ſhare of each would be twice as many as there were maids. And if the terms of this double progreſſion 1, 2, 4, 8, *&c. be counted, as often as there are maids; the laſt term will be the number of ſheep.*

LET $a =$ ſheep, $e =$ maids. Then $\frac{a}{e} = 2e$, and the e^{th} term of the progreſſion 1, 2, 4, 8, &c. $= 2^{e-1}$ (Propor. 25.), therefore $2^{e-1} = a$, *per* queſt. Whence $a = 2ee$, and $2^{e} = 2a$, and expunging a, $2^{e} = 4ee$, or $2^{e-2} = ee$. Therefore $\overline{e-2} \times \text{log: } 2 = 2 \text{ log: } e$, or $.30103e - .60206 = 2 \text{ log: } e$, and $.150515e - \text{log: } e = .30103$. Then to find e (by Rule 5. Prob. xcii.), aſſume $e = 6$, then $.1505e - \text{log:} e = .125$, which ſhould be .301, and the error is $-.176$.

Again, aſſume $e = 7$, then $.1505e - \text{log:} e = .208$, and the error is $-.092$. Then $\frac{1 \times .092}{.176. - .092} = 1.1$; therefore $e = 8.1$ nearly.

Suppoſe $e = 8.1$; then $.1505e - \text{log: } e = .3106$, and the error $= +.0096$. And the correction $= 1.04$, and $e = 8.1 - 1.04 = 7.996$; and $e = 8$ exact, and $a = 128$.

PROB. CCXII.

Two travellers at 150 *miles diſtance ſet out to meet one another. In the ſeveral days,* A *goes at this rate,* 5, 10, 20, 40, *&c.* B *goes* 6, 10, 14, 18, *&c. miles; to find in what time they will meet.*

Let x = days, (by Geom. Prog.) A's laſt day, will be $5\times2^{x-1}$; and his journey $5\times2^{x}-5$. And (Arith. Prog.) B's laſt day is $6+4x-4$ or $4x+2$, and $\frac{12+4x-4}{2}\times x$, or $2x^2+4x$ = B's journey. Whence $2^x\times5-5+4x+2x^2=150$, and $2^x+\frac{4}{10}x^2+\frac{8}{10}x=31$. And $2^x=31-\frac{8}{10}x-\frac{4}{10}xx$. And log: 2^x or $x\times$ log: $2=$ log: $31-\frac{8}{10}x-\frac{4}{10}xx$. By trials x will be found greater than 4; let $n=4$, and $n+v=x$, $b=4$, $c=$ log: 2. Then $cn+cv=$ log: $31-\frac{8}{10}n-\frac{8}{10}v-\frac{4}{10}nn-\frac{8}{10}nv-\frac{4}{10}vv$. But the number belonging to $cn+cv=cn\times:1+mcv+\frac{mcv^2}{2}$ &c. (Prob. lxxxv.) whence $cn+ccmnv+\frac{c^2nmv^2}{2}$ &c. $=31-\frac{8}{10}n-\frac{4}{10}nn-\overline{\frac{8}{10}+\frac{8}{10}n}\times v-\frac{4}{10}vv$.

And reduced,
$$\begin{array}{lll} ccmnv & +\dfrac{ccmn}{2}vv & = 31 \\ +\dfrac{8}{10} & +\dfrac{4}{10} & -\dfrac{8}{10} \\ +\dfrac{8}{10}n & & -\dfrac{4}{10}nn \\ & & -cn \end{array}$$

Which put into numbers, and reverting the ſeries (Prob. lxii), v is had $= .32$; then put new n for $x+v$ or 4.32, and repeat the operation; and at laſt $v = .3256$, and $x = 4.3256$.

PROB. CCXIII.

To find x *in this equation,* $x^x = 123456789$.

Here x will be found between 8 and 9. Put $n=8$, $n+v=x$, $b=\text{log: } n$, $c=\text{log: } 123456789$; then $x\text{log:}x=\text{log:}123456789=c$, or $\overline{n+v}\times\text{log:}\overline{n+v}=c$. But (Prob. lxxxiv.) $\text{log: } n+v = b+\frac{Mv}{n} - \frac{Mv^2}{2n^2}+\frac{Mv^3}{3n^3}-$ &c. and $\overline{n+v}\times\text{log : }\overline{n+v}$

$$\left.\begin{array}{l} = nb + \dfrac{nMv}{n} - \dfrac{nMv^2}{2n^2} + \dfrac{nMv^3}{3n^3}\ \&c. \\ \quad + bv + \dfrac{Mv^2}{n} - \dfrac{Mv^3}{2n^2}.\ \&c. \end{array}\right\} = c;$$

and tranſpoſing and reducing,

$$\overline{\frac{b}{M}+1}\times v + \frac{v^2}{2n} - \frac{v^3}{2.3n^2} + \frac{v^4}{3.4n^3} - \frac{v^5}{4.5n^4}\ \&c. = \frac{c-nb}{M}.$$

And extracting the root (Prob. xciii.) $v = ,64002$, and x or new $n = 8.64002$ for another operation, which will give $x=8.6400268$.

PROB. CCXIV.

To find the value of x in the equation

$$\overline{1000-x}\times\log\colon\overline{1000-x}=x.$$

Put $b=1000$, $x=a+v$, $b-a=g$, $p=\log\colon g$; then $\overline{g-v}\times\log\colon\overline{g-v}=a+v$. And ſubſtituting the logarithmic ſeries inſtead of $\log\colon\overline{g-v}$ (Prob. lxxxiv.), $\overline{g-v}\times\colon p-\frac{v}{g}-\frac{v^2}{2gg}-\frac{v^3}{3g^3}-\frac{v^4}{4g^4}$ &c. $=a+v$, which multiplied and reduced is, $gp-a\overline{-p+2}\,.\,v+\frac{vv}{2g}+\frac{v^3}{6g^2}+\frac{v^4}{12g^3}+\frac{v^5}{4.5g^4}+\frac{v^5}{5.6g^5}$ &c. $=0$. Aſſume $a=836$, and extracting the root (Prob. xciii.) $v=.05315$; and $x=836{,}05315$.

PROB. CCXV.

To find x in the equation

$$\log\colon\overline{1000-x}=\frac{x}{1000-x}-\frac{1000}{x}.$$

Put $n=1000$, $x=a+v$, $g=n-a$, $p=\log\colon g$. Since x is nearly $=860$, aſſume $a=860$; then the equation is $\log\colon\overline{g-v}=\frac{a+v}{g-v}-\frac{n}{a+v}$. But $\log\colon\overline{g-v}=p-\frac{v}{g}-\frac{vv}{2g^2}$ &c. Whence

$$\frac{\overline{a+v}|^2-ng+nv}{\overline{a+v}\times\overline{g-v}}=p-\frac{v}{g}-\frac{vv}{2gg}-\frac{v^3}{3g^3}\ \&c.$$

Which equation reduced gives

$$\begin{array}{llll} agp & -\,3av & +\,\frac{a}{2g}v^2 & +\,\frac{a}{6gg}v^3\ \&c.=0. \\ -\,aa & -\,n & -\,\frac{p}{2} & +\,\frac{1}{2g} \\ +\,ng & +\,gp & & \\ & -\,ap & & \end{array}$$

In numbers,
$4620.3 + 7138v + 3.8702v^2 - .01088v^3 = 0$;
or $1 + 1.543v + .000836v^2 - .0000023v^3 = 0$;
whence, by extracting the root, $v = -,64822$, and $x = 859,35178$.

PROB. CCXVI.

Having given the equations $x^{x+y} = y^n$, *and* $y^{x+y} = x^m$; *to find* x *and* y.

From the first equation $y = x^{\frac{x+y}{n}}$, and from the second, $y = x^{\frac{m}{x+y}}$; therefore $x^{\frac{x+y}{n}} = x^{\frac{m}{x+y}}$. And equating the indices $\frac{x+y}{n} = \frac{m}{x+y}$, and $x + y = \sqrt{mn}$. Whence $y = x^{\sqrt{\frac{mn}{nn}}} = x^{\sqrt{\frac{m}{n}}}$. Therefore by the first equation, $x^{x+x^{\sqrt{\frac{m}{n}}}} = y^n = x^{\sqrt{mn}}$, and again equating their indices, $x + x^{\sqrt{\frac{m}{n}}} = \sqrt{mn}$. Then x being had y is known from the equation $y = x^{\sqrt{\frac{m}{n}}}$.

To find x put $x = v^n$, then $x^{\sqrt{\frac{m}{n}}}$ or $x^{\frac{\sqrt{mn}}{n}} = v^{\sqrt{mn}}$, and $v + v^{\sqrt{mn}} = \sqrt{mn}$. And the root may be extracted by logarithms.

PROB. CCXVII.

To find the value of x *in this equation,* $X^2+X=\frac{1}{x}$, X *being the hyperbolic log: of* x.

Here x is between 1 and 2, therefore put $x=1+v$, then (Prob. lxxxiv. cor. 1.) $X=v-\frac{v^2}{2}+\frac{v^3}{3}-\frac{v^4}{4}$ &c. Whence $\overline{v-\frac{v^2}{2}+\frac{v^3}{3}\,\&c.}\big|^2+v-\frac{v^2}{2}+\frac{v^3}{3}$ &c. $=\frac{1}{1+v}$, and multiplying and reducing $v+\frac{3}{2}v^2-\frac{1}{6}v^3$ &c. $=1$, and by reverſion (Prob. lxii.) $v=.56$, and $x=1.56$.

But becauſe this does not converge faſt enough; put $n=1.56$, and $n+v=x$, $l=.4446858=$ hyp. log: $n=m\times$ log: n; then (Prob. lxxxiv. cor. 2.) $X=l+\frac{v}{n}-\frac{v^2}{2nn}+\frac{v^3}{3n^3}$, whence we ſhall have $\overline{l+\frac{v}{n}-\frac{vv}{2nn}\,\&c.}\big|^2+\overline{l+\frac{v}{n}-\frac{vv}{2nn}}:\times\overline{n+v}=1.$ And when multiplied and reduced,

$$\overline{l+1}\times nl+\overline{\overline{l+1}\big|^2+l}\times v+\frac{l+\frac{3}{2}}{n}vv\ \&c.=1.$$

In numbers,
$1.0021921+2.5318022v+1.246593v^2=1$; or $2.031v+vv=-.0017586$. Whence (Prob. 88.) $v=-.0008661$, and $n+v$ or $x=1.5591339$.

Otherwiſe thus,

Let $l=.4446858$ the h. log: 1.56, or n, as before, $l+s=X$; then the number (x) belonging to $l + s$ or $X = n \times : 1 + s + \frac{1}{2} ss + \frac{1}{6} s^3$ &c. Fig. (Prob. lxxxv.); whence $\overline{l+s}|^2 + l+s : \times n + ns + \frac{1}{2} nss$ &c. $= 1$; and by reduction,

$$\overline{l+1} \times ln + \overline{\overline{l+1}|^2 + l} \times ns + \overline{\tfrac{1}{2}ll + 2\tfrac{1}{2}l + 2} \times ns^2 \text{ \&c.} = 1.$$

In numbers,

$$1.0021921 + 3.949611s + 5.008515ss \text{ \&c.} = 1.$$

$$\text{or } .78858s + ss = -.00043768;$$

And extracting the root (Prob. lxxxviii.) $s = -.0005549$, and $l+s$ or $X = .4441309$, and $\frac{X}{m} = .1928836$ the com. log: x; or elſe $\frac{s}{m} = -.0002410$, and ſince com. log: $1.56 = .1931246$; therefore $.1931246 - .0002410 = .1928836$ the common log: x. Whence $x = 1.559134$.

PROB. CCXVIII.

To find x in the equation $x^{x^x} = 123456789 = b$.

Put $b = 123456789$, and by a few trials you will find x near 2.8, put $n = 2.8$, $n + v = x$, $l = \text{log}: n$. $ml = \text{hyp. log}: n$.

Then (Prob. lxxxv. cor. 6.) $x^x = n^n \times \overline{1 + mlv + v}$. Put $r = n^n$, $e = n^n \times \overline{mlv + v}$, then $x^x = r + e$; let this be an index, then $x^{x^x} = x^{r+e} = \overline{n+v}^{\,r+e} =$ (by the ſame cor.) $n^r \times 1 + mle + \frac{rv}{n} = b$ *per* queſt. Then reſtoring the values of r and e, $n^{n^n} \times : 1 + mln^n$

Fig.

$\times \overline{mlv + v} + \frac{n^n v}{n} : = b.$ Put $g = mln^n \times \overline{ml + 1}$, then $\overline{n}|^{n^n} \times \overline{\underset{n}{1} + gv + n^{n-1}v} = b$, and by reducing, $v = \frac{b - n^{n^n}}{n^{n^n} \times \overline{g + n^{n-1}}}$; here $n^{n^n} = 97620000$, $ml = 1.02962$, $g = 37{\cdot}3368$, $n^{n-1} = 6.3810$, therefore $v = \frac{25830000}{4266000000} = .006054$.

Or let $\frac{b}{n^{n}} = f$. Then $v = \frac{f - 1}{g + n^{n-1}} = \frac{.26466}{43{\cdot}7178} = .006054$; then $n + v$ or $x = 2.806054$ nearly; or put $n = 2.806054$ for another operation.

This problem is eaſily reſolved by rule 5, problem xcii. by making ſeveral ſuppoſitions for the value of x, and finding the correction every time; and ſo you will continually approximate to the true value.

PROB. CCXIX.

If X *be the log.* x, *it is required to find* x, *in the equation* $x^X + X^x = 100$.

Let $n + v = x$, $l = \log{:}n$, $l + s = \log{:}n + v$, $L = \log{:}l$. Then (Prob. lxxxv.) $n + v = n + nms$ &c. whence $\overline{n + nms}|^{l+s} + \overline{l + s}|^{n+nms} = 100$. But (Prob. lxxxv. cor. 6.) $\overline{n + nms}|^{l+s} = n^l \times : \overline{\underline{1 + mls + lms}}$.

And $\overline{l + s}|^{n+nms} = l^n \times 1 + m^2 Lns + \frac{ns}{l}$.

Therefore $nl\times\overline{1+2mls}+l^{n}\times\overline{1+m^{2}Lns+\frac{ns}{l}}=100$.

Or $n^{l}\times 2mls+\overline{l^{n}\times m^{2}Lns+\frac{ns}{l}}=100-n^{l}-l^{n}=d$.

And $s=\frac{d}{2mln^{l}+nLl^{n}m^{2}+nl^{n-1}}$.

To approach nearly to the value of X, we ſhall have X log: x or XX $=$ log: x^{X}, and x log: X $=$ log: X^{x}. Therefore num. of XX $+$ num. of x log: X $=100$. By a few trials X is found between 1.25 and 1.26, but nearer 1.26; therefore ſuppoſe $l=1.257$, then $n=18.072$, $L=.09933$, $n^{l}=38.02$, $l^{n}=62.41$, $d=-.43$, $2mln^{l}=220.1$, $nLl^{n}m^{2}=594.0$, $nl^{n-1}=897.4$, Whence $s=\frac{-.43}{1711.5}=-.000251$, and $X=1.256749$, and $x=18.0613$.

Here we have ſought the logarithm X, for variety; but the number x might have been found, after the manner of the laſt problem.

PROB. CCXX.

Given $x|^{x^{x}}+x^{x}-x+x^{\frac{1}{x}}=200$; *to find* x.

Take n very near the root, to be found by frequent trials, and put $n+v=x$, $l=$log:n, $r=n^{n}$, $f=ml+1$, $p=n^{n^{n}}$, $q=mlrf+\frac{r}{n}$. $t=n^{\frac{1}{n}}$, $a=\frac{m^{l}-1}{nn}$.

Then

Fig. Then $x^{x} = \overline{n+v}|^{n+v} = n^{n} \times \overline{1+mlv+v}$ (Prob. lxxxv. cor. 2.) $= r + frv$.

And $x^{x^{x}} = x^{r+rfv} = \overline{n+v}|^{r+rfv} = n^{r} \times \overline{1 + mlrfv + \frac{rv}{n}} = p + pqv$, (ib. cor. 6.).

Alſo $x^{-x} = \frac{1}{x^{x}} = \frac{1}{r+frv} = \frac{1-fv}{r}$.

And $\frac{1}{x} = \frac{1}{n+v} = \frac{n-v}{nn}$.

Alſo $x^{\frac{1}{x}} = \overline{n+v}|^{\frac{n-v}{nn}} = n^{\frac{1}{n}} \times \overline{1 - \frac{mlv}{nn} + \frac{v}{nn}}$ (ib. cor. 6.) $= t - atv$.

Therefore writing for the ſeveral powers of x, their reſpective values, we have

$$p + pqv + r + rfv + \frac{fv-1}{r} + t - tav = 200.$$

reduced $v = \dfrac{200 - p - r - t + \frac{1}{r}}{pq + rf + \frac{f}{r} - ta}$.

It eaſily appears that x is greater than 2, and trying $2\frac{1}{4}$, it will be found a little too ſmall; therefore aſſume $n = 2.27$, whence there will come out $v = -.0009463$, and therefore $x = 2.2690537$, which may be put for n, for another operation.

SECT. XIV.

Problems of Maxima and Minima.

PROB. CCXXI.

The line AE, *and the two points* B, C, *being given in poſition; to find the point* P, *ſo that* $BP+PC$ *may be the leaſt poſſible.* 145.

TAKE the point p extreamly near P, and draw Bp, Cp, and alſo $pD \perp$ to BP, and $pO \perp$ to CP. Then pD is the increment of BP; and PO the decrement of CP, therefore $DP=OP$, by the nature of the queſtion. And ſince the hypothenuſe Pp is common, $pD=pO$. And $<pPD=pPO$, that is $BPA=CPE$; whence the triangles BAP, CEP, are ſimilar. Put $AE=b$, $AB=p$, $CE=q$, $AP=x$, $EP=b-x$; then AB (p) : AP (x) : : CE (q) : EP $(b-x)$ therefore $qx=pb-px$, and $px+qx=pb$, and $x=\frac{pb}{p+q}$, and $b-x=\frac{bq}{p+q}$.

PROB. CCXXII.

The lines ABC, *and* CE *being given in poſition, and the points* A, B, *being given; to find the point* D *in the line* CE, *where the angle* ADB *is the greateſt poſſible.* 146.

About AB deſcribe a circle to touch the line CE; then the point of contact D is the point required.

For

Fig. 146. For to any other point E, in the line CE, draw AE, BE, and draw BF. Then the angle AEB is leſs than AFB, or its equal ADB (Geom. IV. 13.)

Let BC $=b$, AC $=d$, CD $=x$. Then (Geom. IV. 21. cor. 2.) $xx=bd$, and $x=\sqrt{bd}$.

PROB. CCXXIII.

147. *To draw the ſhorteſt line poſſible, through a given point* P, *placed within the right angle* ABC.

Let CPA be the ſhorteſt line. Draw PD parallel to AB, and PF parallel to CB, and let PD $=b$, PF $=c$, CD $=x$, C$c=e$ an extreamly ſmall quantity, PC $=z$.

By the ſimilar triangles CDP, PFA, $x:z::c:\frac{cz}{x}=$ AP, and by the ſimilar triangles CDP, cHC, $z:x::e:\frac{xe}{z}=$ Hc. Alſo $z:b::e:\frac{be}{z}=$ HC. And by the ſimilar triangles PCH, FGa, $z:\frac{be}{z}::\frac{cz}{x}:\frac{bce}{zx}=a$G. And by the ſimilar triangles CDP, aGA, $x:b::\frac{bce}{zx}:\frac{bbce}{zxx}=$ AG. But H$c=$ AG, that is $\frac{xe}{z}=\frac{bbce}{zxx}$, or $x=\frac{bbc}{xx}$, and $x^3=bbc$, whence $x=\sqrt[3]{bbc}$.

PROB. CCXXIV.

148. *Given the line* EF, *and two points* A, B; *to find the point* D, *ſo that* $a\times$ AD $+\,b\times$ BD, *may be the leaſt poſſible*; a, b *being given numbers.*

Take d infinitely near D, and draw Ad, Bd; on which let fall the perpendiculars Dr, Df. Then will $a\times AD+b\times BD=a\times Ad+b\times Bd$; and by ſubtraction $a\times\overline{AD-Ad}=b\times\overline{Bd-BD}$, or $a\times dr=b\times df$. But in the triangles Ddr, Ddf, the hypothenuſe Dd is common; therefore $dr:df::$ S.dDr : S.dDf : : coſ. ADF : coſ. BDE. Whence $a:b::$ coſ. BDE : coſ. ADF. Fig. 148.

Let AF, BE be perpendicular to EF; and put $AF=c$, $BE=d$, $EF=n$, $DF=x$, $DE=v$. Then DA $(\sqrt{cc+xx}):1::$ DF $(x):\dfrac{x}{\sqrt{cc+xx}}=$ S.DAF $=$ coſ. ADF; then $b:a::\dfrac{x}{\sqrt{cc+xx}}:$ $b\dfrac{ax}{\sqrt{cc+xx}}=$ coſ. BDE $=$ S.DBE. And $b\dfrac{ax}{\sqrt{cc+xx}}:v::1:$ (BD) $\sqrt{aa+vv}$; therefore $v=\dfrac{ax\sqrt{ad+vv}}{b\sqrt{cc+xx}}$, and $bv\sqrt{cc+xx}=ax\sqrt{dd+vv}$. And ſquaring, $bbccvv+bbxxvv=aaddxx+aaxxvv$, but $v=n-x$, put $p=bb-aa$, then $\overline{bbcc+pxx}\times vv=aaddxx$, or $\overline{bbcc+pxx}\times\overline{nn-2nx+xx}=aaddxx$; reduced,

$$px^4-2pnx^3+\begin{array}{l}pnn\\+bbcc\\-aadd\end{array}xx-2nbbccx+b^2c^2n^2=0.$$

PROB. CCXXV.

Three points A, B, C *being given; to find a fourth point* D, *ſo that* $a\times AD+b\times BD+c\times CD$, *may be the leaſt poſſible; where* a, b, c *are given numbers.* 149.

Let D be the point ſought; with radius CD, deſcribe the circle GDH. Take the point d infinitely near D, and draw Ad, Bd; on AD, BD, let fall the perpendiculars dr, df. Then ſuppoſing

Fig. 149. ſing CD to be given, $a \times AD + b \times BD$ will be a minimum. But $a \times Dr$ is the increment of $a \times AD$, and $b \times Df$ is the decrement of $b \times BD$, therefore $a \times Dr = b \times Df$. But in the right-angled triangles D*dr*. D*df*, D*r* : D*f* : : S.D*dr* : S.D*df* : : S.*r*DC, or ADC : S.*mdf* or BDC. Therefore $b : a :: $ S.ADC : S.BDC.

After the ſame manner, ſuppoſing BD given, we ſhall have $c : a ::$ S.ADB : S.BDC. Therefore when $a \times AD + b \times BD + c \times CD =$ minimum; a, b, c, are reſpectively as the ſines of BDC, ADC, ADB; or of BD*m*, AD*m*, BD*r*, which makes 180°. Therefore if a triangle be made of the 3 lines a, b, c; the angles of this triangle will be equal to the angles at D, *viz*, that oppoſite to $a = m$DB, to $b = m$DA, to $c =$ BD*r*. Therefore all the angles about the point D being given; the diſtances AD, BD, CD will be found by Prob. cxxxii.

PROB. CCXXVI.

150. *Given the triangle* ABD, *and the circle* CFK *whoſe center is* A; *to find the point* F *in the circumference* CFK, *that the angle* BFD *may be the greateſt poſſible.*

Through the points B, D, deſcribe the circle BFD to touch the circle CFK in F, the point required. For to any other point C, in the circle CK, draw DC cutting BFD in S, and draw BS, BC. Then the $\angle$BSD or BFD $= \angle$BCD + CBS; therefore BFD is greater than BCD.

On BD let fall the perpendiculars, AH, FI, GE; G being the center of the circle BFD. Then to find its radius GF; let $BE = ED = b$, $HE = c$, $AH = p$, $AF = r$, $GE = x$. Then $AG = \sqrt{\overline{p+x}|^{2} + cc}$

=

Fig. 150. $= \sqrt{pp + cc + 2px + xx}$, and GF $=$ $\sqrt{pp + cc + 2px + xx} - r$, and BG $= \sqrt{bb + xx}$, whence $\sqrt{pp+cc+2px+xx} - r = \sqrt{bb + xx}$, and $\sqrt{pp+cc+2px+xx} = r + \sqrt{bb+xx}$, which ſquared is $cc+pp+2px+xx = rr+bb+xx+2r \times \sqrt{bb+xx}$: Put $s = cc + pp - rr - bb$; then $s + 2px = 2r \times \sqrt{bb+xx}$, and $ss+4spx+4ppxx = 4rrbb + 4rrxx$; reduced,

$$\begin{array}{lcll} 4ppxx & + & 4spx & = 4rrbb. \\ -4rr & & & -ss. \end{array}$$

Then x being found, it will be EG (x) : rad. (1) :: BE (b) : tang. BGE, or its ſupplement BFD.

PROB. CCXXVII.

To find the greateſt area contained under any number of right lines given, and another line unknown. 151.

Let ABCDE be the figure; then ſince ABE + BCDE is a maximum; it is evident, whatever the figure BCDE is, ABE muſt be a right-angled triangle, right-angled at B.

Again, ſince ABC + CDE + ACE is a maximum; it is evident whatever ABC and CDE are, ACE muſt be a right-angled triangle, right-angled at C.

Alſo ſince ABCD + ADE is a maximum; it is plain, whatever the figure ABCD is, ADE muſt be a right-angled triangle, right-angled at D. And ſo on if there were never ſo many lines. And therefore all the angles ABE, ACE, ADE, ſubtended by AE, muſt be right-angles; and conſequently the whole figure is inſcribed in a ſemicircle, whoſe diameter is AE, ſo that the whole may be a maximum.

Therefore if it be required to find the area, we muſt find the diameter AE, and then find the

Fig. the area of the poligon ABCDE inſcribed in a ſemicircle.

PROB. CCXXVIII.

152. *To find a line, which with three given lines, will contain the greateſt area poſſible.*

It is plain the line ſought is the diameter of the ſemicircle in which the three given lines are inſcribed.

Let ABCD, be the quadrangle, draw the diagonals AC, BD, on which let fall the perpendiculars CP, BF.

Let AB $= b$, BC $= c$, CD $= d$, diameter AD $= y$. Then BD $= \sqrt{yy-bb}$, and AC $= \sqrt{yy-dd}$. But (Geom. IV. 28.) CP $= \frac{cd}{y}$, and BF $= \frac{bc}{y}$. Therefore $\overline{b+\frac{cd}{y}}\sqrt{yy-bb} = 2$ area ABCD, and $\overline{d+\frac{bc}{y}}\sqrt{yy-dd} = 2$ area $= \overline{b+\frac{cd}{y}}\sqrt{yy-bb}$, and $\overline{by+cd} \times \sqrt{yy-bb} = \overline{dy+bc}\sqrt{yy-dd}$, and ſquaring and multiplying, $bby^4 + 2bcdy^3 + ccddyy - b^4yy - 2b^3cdy - bbccdd = ddy^4 + 2bcdy^3 + bbccyy - d^4yy - 2bcd^3y - bbccdd$. And reducing

$$\begin{array}{llll} bb\,y^3 & -b^4\;\;y & +2bcd^3 & = 0. \\ -dd & -bbcc & -2b^3cd & \\ & +ccdd & & \\ & +d^4 & & \end{array}$$

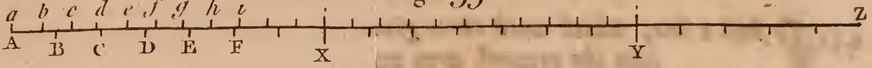
And dividing by $bb-dd$

$$\begin{array}{ll} y^3 & -bb\,y = 2bcd, \\ & -cc \\ & -dd \end{array}$$

and y being known, the area is known from the foregoing ſteps.

PROB.

Fig. 139



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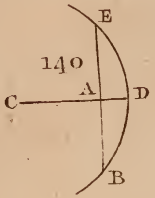

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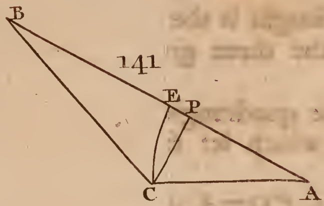

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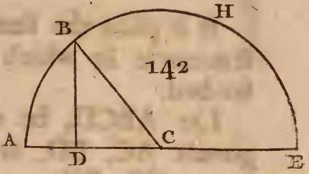

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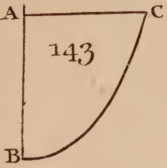

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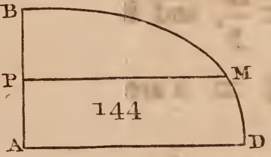

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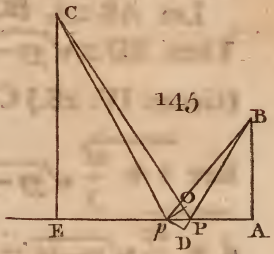

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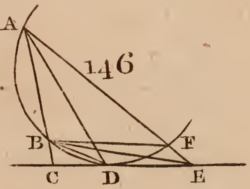

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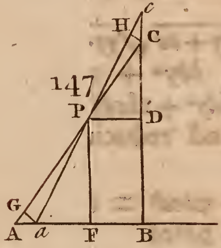

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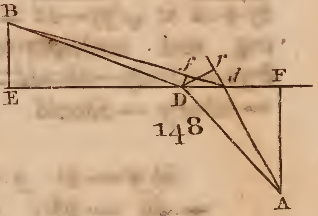

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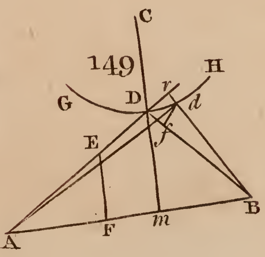

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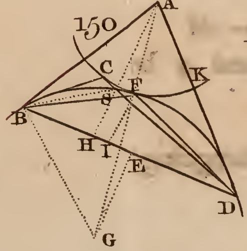

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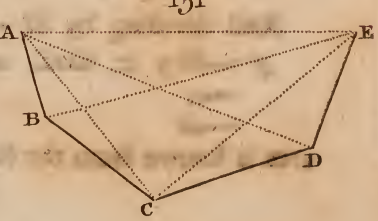


Pl. XII. *pa.* 512

PROB. CCXXIX.

SP *is perpendicular to* PM, *and there is given* SP, SN; *and drawing* NL, *ſo that the angle* LDM *may be equal to* SCP; *to find* CD, *a maximum.* 153.

Draw NA perpendicular to CD, then CA $=$ AD, and CA is a maximum. Put SN $= b$, SP $= d$, SC $= y$, then CN $= b-y$, CP $= \sqrt{yy-dd}$. Then by ſimilar triangles, $y : \sqrt{yy-dd} :: b-y :$ CA $= \frac{b-y}{y}\sqrt{yy-dd} =$ max. and $\overline{b-y}|^2 \times \frac{yy-dd}{yy} =$ max. $= \overline{b-y}|^2 - \frac{dd}{yy} \times \overline{b-y}|^2$. Increaſe y by a very ſmall quantity e, then $\overline{b-y-e}|^2 = \overline{b-y}|^2 - 2e \times \overline{b-y}$. Alſo $\overline{y+e}|^2 = yy + 2ye$, and by diviſion $\frac{dd}{yy+2ye} = \frac{dd}{yy} - \frac{2dde}{y^3}$. Whence $\overline{b-y}|^2 - \frac{dd}{yy} \times \overline{b-y}|^2 = \overline{b-y}|^2 - 2e \times \overline{b-y} - \overline{\frac{dd}{yy} - \frac{2dde}{y^3}} \times \overline{\overline{b-y}|^2 - 2e \times \overline{b-y}}$, and tranſpoſing, $2e \times \overline{b-y} = \frac{2dde}{y^3} \times \overline{b-y} + \frac{2dde}{yy} \times \overline{b-y}$, and dividing by $2e \times \overline{b-y}$, $1 = \frac{dd}{y^3} \times \overline{b-y} + \frac{dd}{yy}$, and multiplying by y^3, we have $y^3 = ddb - ddy + ddy$, or $y^3 = bdd$, and $y = \sqrt[3]{bdd}$.

PROB. CCXXX.

154. *Given the ſituation of the two places* A, E, *and the river* BD; *and ſuppoſe a traveller going from* A *to* C, *can travel* 6 *miles an hour on this ſide the river from* A *to* C; *and* 9 *miles an hour on the other ſide from* C *to* E; *it is required to know where he muſt croſs the river* BD, *ſo that he may go from* A *to* E *in the leaſt time poſſible.*

Let AB, ED be perpendicular to BD; let $AB=a$, $DE=b$, $BD=d$, $m=6$, $n=9$, $BC=x$. Then $CD=d-x$, $AC=\sqrt{aa+xx}$, $CE=\sqrt{bb+\overline{d-x}|^2}$. And *per* queſt. $\overset{mil.}{m} : \overset{ho.}{1} :: \sqrt{aa+xx} : \dfrac{\sqrt{aa+xx}}{m}$ $=$ time in AC, and $\overset{mil.}{n} : \overset{ho.}{1} :: \sqrt{dd+\overline{d-x}|^2} : \dfrac{\sqrt{bb+\overline{d-x}|^2}}{n}$ $=$ time in CE. Therefore $\dfrac{\sqrt{aa+xx}}{m} + \dfrac{\sqrt{bb+\overline{d-x}|^2}}{n} =$ minimum. Or $n\sqrt{aa+xx}+m\sqrt{bb+\overline{d-x}|^2} =$ min. Write $x+e$ for x; then $xx = xx+2xe$, and $\overline{d-x-e}|^2 = \overline{d-x}|^2 - 2e \times \overline{a-x}$. Therefore we have $n\sqrt{aa+xx+2xe} + m\sqrt{bb+\overline{d-x}|^2 - 2e \times \overline{d-x}}$ $= n\sqrt{aa+xx} + m\sqrt{bb+\overline{d-x}|^2}$. But $\sqrt{aa+xx+2xe} = \sqrt{aa+xx} + \dfrac{xe}{\sqrt{aa+xx}}$, and $\sqrt{bb+\overline{d-x}|^2 - 2e \times \overline{d-x}} = \sqrt{bb+\overline{d-x}|^2} -$

$\frac{e\times\overline{d-x}}{\sqrt{bb+\overline{d-x}|^2}}$. Therefore $n\sqrt{aa+xx}+\frac{nxe}{\sqrt{aa+xx}}$ Fig. 154.
$+m\sqrt{bb+\overline{d-x}|^2}-\frac{me\times\overline{d-x}}{\sqrt{bb+\overline{d-x}|^2}}=n\sqrt{aa+xx}$
$+m\sqrt{bb+\overline{d-x}|^2}$. Therefore $\frac{nxe}{\sqrt{aa+xx}}-\frac{me\times\overline{d-x}}{\sqrt{bb+\overline{d-x}|^2}}=0$. And multiplying, $nx\sqrt{bb+dd-2dx+xx}=\overline{md-mx}\sqrt{aa+xx}$; and ſquaring $nnbbxx+nnddxx-2nndx^3+nnx^4=mmddaa-2mmdaax+mmaaxx+mmddxx-2mmdx^3+mmx^4$. And being reduced is,

$$\begin{array}{lllll} nnx^4 & -2nndx^3 & +nnddxx & +2mmdaax & -m^2d^2a^2=0. \\ -mm & +2mmd & +nnbb & & \\ & & -mmdd & & \\ & & -mmaa & & \end{array}$$

PROB. CCXXXI.

Within the given angle ACB, *to cut off a given area with the ſhorteſt line* AB. 155.

Let the area $=b$; s, $c=$ ſine and coſ C; CA $=x$, CB $=y$, then *per quæſt.* $sxy=b$, and by Trigonometry AB $=\sqrt{xx+yy-2cxy}=$ min.; therefore $xx+yy-2cxy=$ min. but $xy=\frac{b}{s}$, and $y=\frac{b}{sx}$, therefore $xx+\frac{bb}{ssxx}-\frac{2cb}{s}=$ min. or $xx+\frac{bb}{ssxx}=$ min. Put $x+e$ for x, then $\overline{x+e}|^2=x^2+2ex$, and $\frac{1}{\overline{x+e}|^2}$ or $\overline{x+e}|^{-2}=\frac{1}{xx}-\frac{2e}{x^3}$. Whence

Fig. 155. $xx + 2ex + \frac{bb}{ss} \times \frac{1}{xx} - \frac{bb}{ss} \times \frac{2e}{x^3} = xx + \frac{bb}{ssxx}$, and $2ex - \frac{2bbe}{ssx^3} = 0$, and $x = \frac{bb}{ssx^3}$, whence $x^4 = \frac{bb}{ss}$. But $y^4 = \frac{b^4}{s^4x^4} = \frac{b^4}{s^4} \times \frac{ss}{bb} = \frac{bb}{ss}$; therefore $y^4 = x^4$, and $y = x = \sqrt{\frac{b}{s}}$.

PROB. CCXXXII.

156. *To find the greateſt parallelogram inſcribed in a triangle.*

Let the parallelogram BDEF be inſcribed in the triangle ABC. Put $AB = a$, $BC = b$, $DB = x$, $DE = y$. Then by the ſimilar triangles ABC, ADE, $a - x : y :: a : b$, and $ay = ba - bx$, and $y = \frac{ba-bx}{a} = b - \frac{bx}{a}$. But $xy = max$, or $bx - \frac{bxx}{a} =$ max. Put e for the ſmall increment of x, then the increment of bx is be, and the decrement of xx is $\overline{x+e}|^2 - xx = 2xe$, and the decrement of $\frac{bxx}{a} = \frac{2bxe}{a}$, whence $be = \frac{2bxe}{a}$, and $1 = \frac{2x}{a}$, and $x = \frac{1}{2}a$. Therefore $y = \frac{1}{2}b$.

PROB. CCXXXIII.

157. *Given the point* P *within the right angle* ACB; *to draw the line* APB, *ſo that* AP × PB *may be a minimum.*

Draw DP, PF parallel to CB, CA; and put $CF = b$, $CD = c$, $AD = x$. Then by ſimilar triangles

angles $x : b :: c : y$, and $xy = bc$, and $y = \frac{bc}{x}$. Fig. 157. Then $AP = \sqrt{bb+xx}$, and $PB = \sqrt{cc+yy}$; and $AP \times PB = \sqrt{bb+xx} \times \sqrt{cc + \frac{bbcc}{xx}} =$ min. and ſquaring, $bbcc + \frac{b^4cc}{xx} + ccxx + bbcc =$ min. and $ccxx + \frac{b^4cc}{xx} =$ min. Whence $cc \times \overline{x+e}|^2 + \frac{b^4cc}{\overline{x+e}|^2} = ccxx + \frac{b^4cc}{xx}$, or $cc \times \overline{xx+2xe} + b^4cc \times \overline{\frac{1}{xx} - \frac{2e}{x^3}} = ccxx + \frac{b^4cc}{xx}$. Whence $2ccxe - \frac{2b^4cce}{x^3} = 0$, $x = \frac{b^4}{x^3}$, or $x^4 = b^4$, and $x = b$, whence $y = c$. And $AC = x + c = b + c$, and $CB = b + y = b + c$. Therefore $AC = CB$.

And if it be required to have AP+PB, *a minimum*, we ſhall have $\sqrt{bb+xx} + \sqrt{cc+yy} =$ min. or $\sqrt{bb+xx} + \sqrt{cc + \frac{bbcc}{xx}} =$ min. But $\sqrt{bb + \overline{x+e}|^2} - \sqrt{bb+xx} = \frac{xe}{\sqrt{bb+xx}} =$ the increment of $\sqrt{bb+xx}$. And in like manner $\frac{-bbcce}{x^3\sqrt{cc+\frac{bbcc}{xx}}}$ is the increment of $\sqrt{cc+\frac{bbcc}{xx}}$, or $\frac{bbce}{xx\sqrt{xx+bb}}$ its decrement. Therefore $\frac{xe}{\sqrt{bb+xx}} = \frac{bbce}{xx\sqrt{bb+xx}}$, and $x^3 = bbc$, or $x = \sqrt[3]{bbc}$, as in Prob. ccxxiii. by another method

PROB. CCXXXIV.

Given the ſum of the legs of a right-angled triangle; to find the legs, ſo as to contain the greateſt area poſſible.

Let a = ſum of the legs, x = one of them; then $x \times \overline{a-x} = 2$ area = max. therefore $\overline{x+e} \times \overline{a-x-e} = x \times \overline{a-x}$, that is, $ax - xx - xe + ae - xe = ax - xx$, and $ae - 2xe = 0$, or $2x = a$, whence $x = \frac{1}{2}a$, and $a - x = \frac{1}{2}a$. Therefore the legs are equal. And therefore when the area is given; the ſum of the legs will be the leaſt, when they are equal.

PROB. CCXXXV.

Given the area of a right angled triangle; to find the ſides, when the perimeter is the leaſt poſſible.

Let a = area, x = ſum of the legs, v, y = the two legs; then $vv + yy + 2vy = xx$, but $vy = 2a$, and $vv + yy = xx - 2vy = xx - 4a$, and the hypothenuſe $= \sqrt{vv+yy} = \sqrt{xx-4a}$; therefore $x + \sqrt{xx-4a}$ = perimeter = min. write $x+e$ for x; then $\sqrt{\overline{x+e}^2 - 4a} = \sqrt{xx + 2x - 4a} = \sqrt{xx-4a} + \frac{xe}{\sqrt{xx-4a}}$; whence $x + e + \sqrt{xx-4a} + \frac{xe}{\sqrt{xx-4a}} = x + \sqrt{xx-4a}$, and $e + \frac{xe}{\sqrt{xx-4a}} = 0$. And $e\sqrt{xx-4a} = -xe$, and $eexx - 4aee = xxee$, and $4aee = 0$, and $ee = \frac{0}{4a}$, or $e = 0$. And therefore ſince the increment of x is nothing; therefore x is a minimum, and when x or the ſum is a minimum; then the legs are equal, by the laſt problem; therefore $v = y = \frac{1}{2}x$, and $\frac{1}{4}xx = 2a$, or $x = \sqrt{8a}$. Fig.

PROB. CCXXXVI.

Given the ſolidity of a ſquare pyramid DF; *to find the ſlant ſide* AB *the leaſt poſſible.* 158.

Let b = ſolidity, x = CB the height, $y = 2$AC the breadth, then AB $= \sqrt{xx + \frac{1}{4}yy}$. But $\frac{1}{3}xyy = b$, and $yy = \frac{3b}{x}$; therefore AB $= \sqrt{xx + \frac{3b}{4x}} =$ minimum, and $xx + \frac{3b}{4x} =$ min. Put $x + e$ for x, then $xx = xx + 2xe$, and $\frac{3b}{4x} = \frac{3b}{4x+5e} = \frac{3b}{4x} - \frac{3be}{4xx}$; whence $xx + 2xe + \frac{3b}{4x} - \frac{3be}{4xx} = xx + \frac{3b}{4x}$, and $2xe - \frac{3be}{4xx} = 0$, and $8x^3 = 3b$, whence $x = \sqrt[3]{\frac{3b}{8}} = \frac{1}{2}\sqrt[3]{3b}$.

PROB. CCXXXVII.

Given the ſolidity of the ſquare pyramid DF, *to find that which has the leaſt ſurface, excluding the baſe.* 158.

Let b = ſolidity, x = CB the height, $y = 2$AC, or 2AD the breadth. Then AB $= \sqrt{xx + \frac{1}{4}yy}$, and $\frac{1}{2}y \times \sqrt{xx + \frac{1}{4}yy} =$ DBL, and $2y\sqrt{xx + \frac{1}{4}yy} =$ ſurface. But $\frac{1}{3}xyy = b$, and $yy = \frac{3b}{x}$. Whence the ſurface $= 2\sqrt{\frac{3b}{x}} \times \sqrt{xx + \frac{3b}{4x}} = 2\sqrt{3bx + \frac{9bb}{4xx}}$ = maxi-

Fig. 158. = maximum. And $3bx + \frac{9bb}{4xx}$ = maximum, or $12bx + \frac{9bb}{xx}$ = max. write $x + e$ for x, then $\frac{9bb}{xx} = \frac{9bb}{xx + 2xe} = \frac{9bb}{xx} - \frac{18bbe}{x^3}$. Therefore $12bx + 12be + \frac{9bb}{xx} - \frac{18bbe}{x^3} = 12bx + \frac{9bb}{xx}$; and $12be - \frac{18bbe}{x^3} = 0$, or $12b = \frac{18bb}{x^3}$, and $12x^3 = 18b$, and $x^3 = \frac{3b}{2}$, whence $x = \sqrt[3]{\frac{3}{2b}}$.

PROB. CCXXXVIII.

159. *To find the greateſt cylinder, inſcribed in a given cone.*

Let axis $AB = a$, BC or $BF = b$, $c = 3.1416$, $DB = x$, DE or $DG = y$. Then by the ſimilar triangles ABC, ADE, $a - x : y :: a : b$, and $ay = ba - bx$, or $bx = ab - ay$ and $x = \frac{ab - ay}{b}$. But $cyyx$ = maximum, or $cyy \times \frac{ab - ay}{b} = \frac{abcyy - acy^3}{b}$ = max. that is $\frac{ac}{b} \times \overline{byy - y^3}$ = max and $byy - y^3$ = max. put $y + e$ for y, then byy becomes $byy + 2bye$, and y^3 becomes $y^3 + 3y^2e$. Whence $byy + 2bye - y^3 - 3y^2e = byy - y^3$. And $2bye - 3yye = 0$, or $2by = 3yy$, and $y = \frac{2}{3}b$. Whence $x = \frac{ab - \frac{2}{3}ab}{b} = \frac{1}{3}a$.

Fig.

PROB. CCXXXIX.

Given the weights of two elaſtic bodies A, C; *to find the weight of the intermediate body* B; *ſo that* A *ſtriking* B *at reſt, and* B *with the motion acquired, ſtriking* C *at reſt, may make* C*'s motion the greateſt poſſible.*

Let $x =$ weight of B, $a =$ velocity of A.
$y =$ velocity of B, acquired by the ſtroke.
$v =$ velocity of C, by the ſtroke.

Then Aa is the motion of both A and B, after the ſtroke, as well as before; and a is the difference of their velocities; therefore $y - a$ is the velocity of A after the ſtroke. And ſince the ſum of their motions remains the ſame (Mechan. 10.), therefore $xy + \overline{y - a} \times A = Aa$, or $xy - Aa + Ay = Aa$, that is, $xy + Ay = 2aA$, and $y = \dfrac{2aA}{A + x}$.

Again, xy is the motion of both B and C, and y the difference of their velocities, as well after as before the ſtroke of B. Therefore $v - y$ is the velocity of B after its ſtriking C. Whence $Cv + \overline{v - y} \times x = xy$, or $Cv + xv = 2yx$. Whence $v = \dfrac{2yx}{C + x} = \dfrac{4aAx}{\overline{A + x} \times \overline{C + x}} =$ maximum *per* queſtion; or $\dfrac{x}{\overline{A + x} \times \overline{C + x}} =$ maximum; or $\dfrac{\overline{A + x} \times \overline{C + x}}{x} =$ minimum, that is,

$$\frac{AC + \overline{A + C}.x + xx}{x} = \text{minimum, or } \frac{AC}{x} + A + C$$

Fig. $A + C + x =$ minimum; therefore $\frac{AC}{x} + x =$ minimum; put $x + e$ for x; then $\frac{AC}{x + e} = \frac{AC}{x} - \frac{ACe}{xx}$, and therefore $\frac{AC}{x} - \frac{ACe}{xx} + x + e = \frac{AC}{x} + x$, and throwing out the ſuperfluous quantities, $e - \frac{ACe}{xx} = 0$, and $xx = AC$, whence $x = \sqrt{AC}$.

PROB. CCXL.

To find $x^m - x^n$ the greateſt poſſible, ſuppoſing n greater than m.

Write $x + e$ for x, then $\overline{x+e}|^m = x^m + m^{m-1}e$, and $\overline{x+e}|^n = x^n + nx^{n-1}e$; therefore $\overline{x+e}|^m - \overline{x+e}|^n = x^m - x^n$; that is, $x^m + mx^{m-1}e - x^n - nx^{n-1}e = x^m - x^n$. And by ſubtraction, $mx^{m-1}e - nx^{n-1}e = 0$, or $mx^{m-1} = nx^{n-1}$, or $mx^m = nx^n$. And n being greater than m, we have $nx^{n-m} = m$, and $x^{n-m} = \frac{m}{n}$; whence

$$x = \sqrt[n-m]{\frac{m}{n}}.$$

Fig.

PROB. CCXLI.

To find the greateſt parallelogram inſcribed in the given curve AMC. 160.

Let MPBF be the greateſt parallelogram. To the point M where it touches the curve, draw the tangent TMD. Then if the ſubtangent PT be equal to the height of the parallelogram PB, then MPBF is the greateſt parallelogram. For it is plain from Problem ccxxxii, that this parallelogram is the greateſt that can be inſcribed in the triangle TDB; and as this is greater than any other that can be inſcribed in the triangle, ſo, much more, is it greater than any other that can be inſcribed in the curve, ſince the angle M which is in the curve, will in all other caſes fall ſhort of the tangent.

Therefore knowing the method of drawing a tangent to the curve; you muſt ſeek the point P, where the ordinate PM being erected, and the tangent TM drawn, TP may be equal to PB. Thus if AM be a parabola; put $AB = a$, $AP = x$, then by the nature of the curve, $AF = x$, whence $TP = 2x$, $PB = a - x$, therefore $2x = a - x$, $3x = a$, or $x = \frac{1}{3}a$.

And the ſame will hold good, if not in all, yet in moſt curves which are convex to the axis. For ſince the parallelogram is the greateſt for the triangle, it will alſo be greateſt for the curve, ſince the curve at that place coincides with the tangent.

Otherwiſe thus,

Suppoſe the nature of the curve be $rx^m = y^n$, where $AP = x$, $PM = y$, alſo $AB = a$, $BC = b$,

Then

Fig. 160. Then PB $= a - x$, and $\overline{a-x} \times y =$ max. But $y = r^{\frac{1}{n}} x^{\frac{m}{n}}$, therefore $\overline{a-x} \times r^{\frac{1}{n}} x^{\frac{m}{n}} =$ max. or $ax^{\frac{m}{n}} - x^{\frac{m+n}{n}} =$ maxim. put $x + e = x$; then $a.\overline{x+e}|^{\frac{m}{n}} = ax^{\frac{m}{n}} + \frac{m}{n} ax^{\frac{m-n}{n}} e$, and $\overline{x+e}|^{\frac{m+n}{n}} = x^{\frac{m+n}{n}} + \frac{m+n}{n} x^{\frac{m}{n}} e$. Therefore

$$ax^{\frac{m}{n}} + \frac{m}{n} ax^{\frac{m-n}{n}} e - x^{\frac{m+n}{n}} - \frac{m+n}{n} x^{\frac{m}{n}} e = ax^{\frac{m}{n}} - x^{\frac{m+n}{n}};$$

whence $\frac{m}{n} ax^{\frac{m-n}{n}} e - \frac{m+n}{n} x^{\frac{m}{n}} e = 0$, or $max^{\frac{m-n}{n}} = \overline{m+n} \times x^{\frac{m}{n}}$; and dividing by $x^{\frac{m-n}{n}}$, $ma = \overline{m+n} \times x^{\frac{m-m+n}{n}} = \overline{m+n} \times x$, whence $x = \frac{ma}{m+n}$. Which is general for all parabolical figures. Thus if $m=1$, $n=2$, as in the common parabola, then $x = \frac{1}{3}a$, and if $m = 2$, $n = 1$, then is $x = \frac{2}{3}a$, as in the ſame parabola, with its convexity towards the axis. If $m = 1$, $n = 1$; then $x = \frac{1}{2}a$, for the triangle, as was proved before.

PROB. CCXLII.

161. *Given the diſtance of the point* A *from the perpendicular plane* BC; *to find the poſition of the plane* AC, *through which a body ſhall deſcend in the ſhorteſt time poſſible to the plane* BC.

Let AB be perpendicular to BC, AD parallel to it, and CD perpendicular to AC. Put AB$=b$, BC$=x$; then (Mechan. prop 34. cor. 1. 4to.) in the

time a body deſcends thro' the inclined plane AC, Fig.
another body will fall perpendicularly through the 161.
ſpace AD. Therefore as the time in AC muſt be a minimum, the time in AD muſt be a minimum, and AD itſelf muſt be a minimum. By the ſimilar triangles BAC, CAD, it is BC (x) : CA $(\sqrt{bb+xx})$: : CA $(\sqrt{bb+xx})$: AD $= \frac{bb+xx}{x} =$ minimum. And $\frac{bb}{x} + x =$ min. write $x+e$ for x, then $\frac{bb}{x+e} = \frac{bb}{x} - \frac{bbe}{xx}$; therefore $\frac{bb}{x} - \frac{bbe}{xx} + x + e = \frac{bb}{x} + x$, and $-\frac{bbe}{xx} + e = 0$, or $1 = \frac{bb}{xx}$, and $xx = bb$, or $x = b$, therefore BC = BA.

Otherwiſe thus,

Deſcribe the circle AGC with the center B, and 162.
radius BA; draw AC and any other line AE, and CGF parallel to it. Then (Mechan. prop. 37. cor. 1. 4to.) the times of a body's deſcending through GC, AC, are equal. And the times of deſcending through the equal lines, of equal inclinations, AE, FC are equal. But the time of deſcending through GC is leſs than the time of deſcending through FC. Therefore the time of deſcending through AC is leſs than the time of deſcending through any other line AE.

PROB. CCXLIII.

AB *is a horizontal line,* BD *an inclined plane. It is* 163.
required to find the poſition of the plane AD, *through which a body deſcending from* A *ſhall arrive at the plane* BD, *in the leaſt time poſſible.*

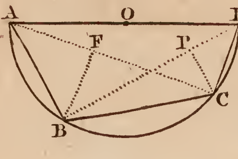
Suppoſe AD to be the plane, draw AL perpendicular to AB, and DH perpendicular to AL, and

Fig. 163. and DF perpendicular to AD. And put $b=$AB, s, $c=$ ſine and coſ. B, BD $=x$, AD $=y$. Then by plain Trigonometry $\sqrt{bb+xx-2cbx}=y$; and AD (y) : S.B (s) : : BD (x) : $\frac{sx}{y}$ = S.BAD or ADH. And rad (1) : AD (y) : : S. ADH $\left(\frac{sx}{y}\right)$: AH $=sx$. And by ſimilar triangles, AH (sx) : AD (y) : : AD (y) : AF $=\frac{yy}{sx}=\frac{bb+xx-2cbx}{sx}$. But the time of falling through AF is equal to the time of deſcending through AD (Mech. prop. 34. cor. 1. 4to.). And this time is a minimum, therefore AF is a minimum, that is, $\frac{bb+xx-2cbx}{sx}=$ min. and $\frac{bb+xx-2cbx}{x}=$ min. or $\frac{bb}{x}+x-2cb=$ min. whence $\frac{bb}{x}+x=$ min. as in the laſt problem. And therefore $x=b$, or AD = AB.

Or thus,

On AF deſcribe a ſemicircle ADF to touch the line AL in D; draw AD, which will be the line of ſhorteſt time. For the time of deſcending through all the cords in the ſemicircle will be equal (Mechan. prop. 37. cor. 1. 4to.) to the time in AD. But the time in any cord is ſhorter than the time in the ſame chord when produced to the line BL, which lies without the circle. And therefore the time in AD is alſo ſhorter than in any other line drawn to BL.

Fig. 152.



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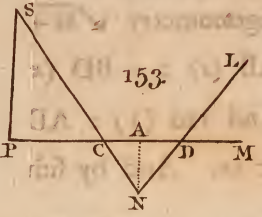

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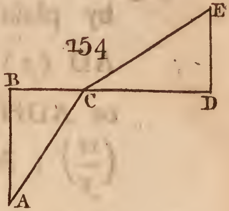

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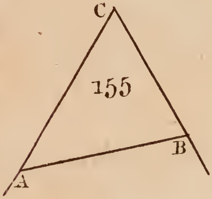

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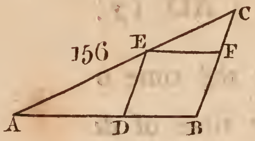

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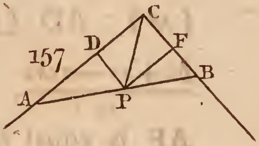

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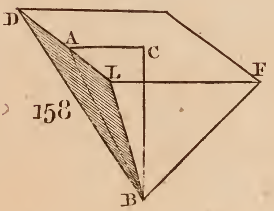

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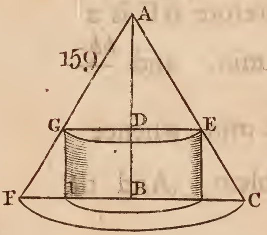

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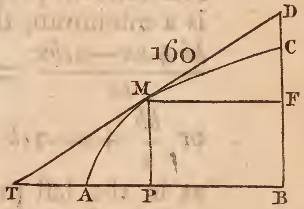

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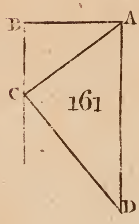

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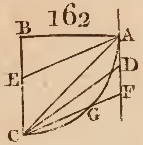

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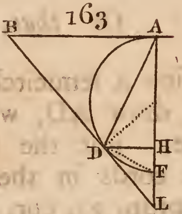

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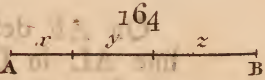

Pl.XIII. *the End.*

PROB. CCXLIV.

To divide a given line AB, *into three parts*, x, y, z; *ſo that* $xyyz^3$ *may be the greateſt product poſſible.* 164.

Firſt, ſuppoſe $x+y=b$ a given quantity, to find $xyy=$ a maximum. Then $x=b-y$, and $\overline{b-y}\times yy$ or $byy-y^3=$ max. put $y+e$ for y, then $b\times\overline{y+e}|^2-\overline{y+e}|^3=$ max. that is, $byy+2bye-y^3-3yye=byy-y^3$, and $2bye-3yye=0$, and $3y=2b$, or $y=\frac{2}{3}b$, and $x=\frac{1}{3}b$. Therefore $y=2x$.

Again, let $x+y+z=d$, to find $xy^2z^3=$ max. Then by what is gone before, whatever z be, y will be $=2x$. Whence $4x^3z^3=$max. But $x+y=d-z$, or $3x=d-z$, and $4x^3z^3=\frac{4}{3}\times\overline{d-z}|^3\times z^3=$max. and $\overline{d-v}|^3\times z^3=$max. or $\overline{d-z}\times z=$ max. or $dz-zz=$ max. put $z+e$ for z, then $dz+de-\overline{z+e}|^2=dz-zz$, or $dz+de-zz-2ze=dz-zz$, or $de-2ze=0$; whence $2z=d=x+y+z$, and $z=x+y=3x$. Therefore $x+2x+3x$ or $6x=d$, and $x=\frac{1}{6}d$, and $y=(2x=)\,\frac{2}{6}d$, and $z=(3x=)\,\frac{3}{6}d$.

POST-

POSTSCRIPT.

HAVING met with an account of my book of Algebra in the Monthly Review, I intend to make a few Remarks of what has been ſaid, that the reader may not be miſled by the trifling Objections of ignorant Pretenders, that ſet up to be judges in things they have no more than a ſuperficial knowledge of.

In commenting upon the 73d Prob. pag. 209. the Critic begins thus; *Among the Problems in the ſeventh Section, is one of ſo extraordinary a nature, that we ſhall beg leave to lay it before the reader entire.* Now I cannot think why this Prob. is ſaid to be of an *extraordinary Nature*, except it be that the Critic has not met with ſuch a thing in all his reading before. But I can aſſure the reader, that there is not one of theſe things mentioned in that Problem, but what he will frequently meet with in practice, if his deſign is to uſe himſelf to Calculations, and to the ſolving of Problems. And, I believe, I ſhould have been guilty of a great overſight, and therefore ſhould have done a manifeſt injury to my reader, if I had not given him ſome light in this affair, and ha' laid down ſome rules how to manage ſuch expreſſions when they occur, rather than leave him in the dark without at all explaining their nature.

After I had given ſuch rules and obſervations as I thought proper; I tell the reader in the Scholium annext, that 0 in a mathematical ſenſe never ſignifies abſolute nothing, but always nothing in reſpect to the object under conſideration. To which this Objector replies, *The above elucidation, however true in that particular, is not, we apprehend, ſufficient to remove the difficulties that attend this Problem. For ſuppoſe inſtead of our being employed in conſidering the area of a ſuperficies, our attention had been engaged in conſidering the length of a line. It will then ſurely follow, that when its length vaniſhes, it becomes a mathematical point or nothing.* I know not what ſort of elucidation may be ſufficient to remove difficulties out of ſome peoples heads. But it will ſurely follow, that this Objector does not know what he is talking about. For the very inſtance he brings is ſufficient to confute him. For when the length of the line is vaniſhed, it then *becomes a mathematical point, or nothing*; that is, it becomes nothing when compared to a line. But will he ſay that a geometrical point is abſolutely nothing? If ſo, it would be impoſſible for Geometers to give a definition of a point as a geometrical term, becauſe abſolute nothing has no manner of exiſtence, and cannot be a term in any ſcience. I ſaid a line is nothing when compared to a ſurface, though it is ſomething in itſelf. And for the ſame reaſon, a point is nothing when compared to a line, and yet is a thing real in itſelf, and not abſolutely nothing.

He proceeds, *But we cannot compare mathematical points together*, BECAUSE *they are totally deſtitute of parts*; *and without parts there can be no compariſon.* One would think that things are the moſt eaſily compared, when they are under equal circumſtances, and of the ſame kind. For points muſt either be equal or elſe unequal to one another; and I take equality to denote a compariſon. But if one be *deſtitute of parts*, and the other has parts, there's an end of any compariſon among theſe, as being different kinds of things. But being *deſtitute of parts* is not the reaſon why things cannot be compared. For example, ſuppoſe any one ſhould try to compare a line with a ſurface, and give this as a reaſon that they cannot be compared, *viz.* that both of them are deſtitute of parts; I think he muſt expect to be laught at for aſſigning ſo odd a reaſon, ſince the true reaſon is, that they are of different kinds; and for that reaſon no compariſon can be made. But whether we can or we cannot compare mathematical points together, is a thing I have nothing to do with; but Dr. Halley has taken ſome pains to ſhew the proportion of mathematical points to one another. And I leave him and this Critic to determine the matter between them.

He goes on, *beſides* (ſays he) *we have often Equations where* o *ſignifies abſolute nothing. Thus, if* $x=y$, *then* $x-y=0$. Yes, in abſtracted Equations made at pleaſure among letters that have no meaning, or which relate to nothing; and then it is no wonder if it is ſo. That is, when his Problem is about nothing at all, the reſult may be abſolutely nothing. But I was writing about practical things; and any body may underſtand, that when a man is ſolving a Problem in earneſt, it is always in regard to ſome ſort of quantity, and then his o will always be of the ſame kind as the quantity concerned, and has relation to nothing elſe. Even in his Equation $x - y = 0$, x and y if they be any thing, muſt be quantities of the ſame kind, or no ſubtraction can be made; and then his o will unavoidably be of the ſame kind too.

What follows is very *extraordinary*; he ſays, *multiplication is nothing more than a number of additions, and diviſion a number of ſubtractions. Conſequently, if we can neither augment nor leſſen a quantity by the addition or ſubtraction of* o; *we can neither augment nor leſſen it by the multiplication or diviſion of* o. *For otherwiſe the very baſis of Arithmetic would be deſtroy'd*, &c. That is, for example, becauſe $0 + 9 = 9$, therefore 0×9 muſt alſo be $= 9$, *for otherwiſe the very baſis of Arithmetic will be deſtroy'd*. A moſt excellent reaſon ſurely to deſtroy the principles of Arithmetic. Such ſort of Critics as theſe, have I the luck to be concern'd with.

Then he proceeds; *In fact, the Cypher is* ONLY *the limit or boundary between negative and affirmative quantities the point from which both begin; and through which they muſt paſs in order to change their denomination.* This Objector then ſeems to know nothing of the great uſe of the o in all arithmetical Operations. And as little has he been uſed to the ſolving Problems; or elſe he would ha' known what frequently happens, that o may be one of the roots of an equation; and then o has as real, intelligible, determinate a value as 1, 2, 3, &c. and this is ſomething more than being the mere boundary between negative and affirmative quantities; about which the Algebraiſt troubles not his head, but attends only to the real uſe of that root when it ſo happens.

At laſt he concludes, that *the difficulties attending the Ideas of infinity and nothing, ought rather to be imputed to the folly of comparing things of different kinds.* But this will not do, ſince no body that ever diſputed about theſe things, was ever ſo ignorant, as to think, that there can be any compariſon made, by way of equality, among heterogeneous quantities. For inſtance, whoever was ſo fooliſh as to ſay, that ſuch a quantity of ſpace was equal to ſo much ſolidity, or equal to ſuch a length of a line. This therefore is not the reaſon why the ideas of o and infinity are ſo difficult to us. For my part I know no difficulty at all of comprehending them in a mathematical ſenſe, and that's all I have to do with them as a Mathematician.

I ought to take notice for the ſake of the reader, that he has made nonſenſe of the 5th Corollary. The true reading is, *Hence alſo* $0^0 = 1$, *or the infinitely ſmall power, of an infinitely ſmall quantity, is infinitely near* 1.

The very next Prob. to this is of the like kind; which is, to find the proportion of o's in particular caſes. This he has not thought fit to meddle with, tho' of the ſame *extraordinary nature*, altho' it is rather more exceptionable than the former. The reaſon he does not mention it, I ſuppoſe is, becauſe ſeveral of the moſt eminent Geometers have handled it, and he might be call'd to account for contradicting them; and therefore he chooſes to let it paſs without any notice, not even ſo much as to tell his readers, that he never ſaw it ſolved by common Algebra before.

Thus I think I have anſwered all his Objections, and ſhewn, that what I have aſſerted in that Prob. is ſolid and true, and that his Objections are mere Cavils, and amount to *abſolutely nothing*.

W. E.

F I N I S.

ERRATA.

In Plate VI. Fig. 54, for A *read* B, for B *read* C, for C *read* A.

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